

TENSOR DIAGRAMS AND CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper, we describe a class of elements in the ring of $\mathrm{SL}(V)$ -invariant polynomial functions on the space of configurations of vectors and linear forms of a 3-dimensional vector space V . These elements are determined by Chebyshev polynomials of the first and second kind with coefficients. We also investigate the relation between these polynomials and Lusztig's dual canonical basis in tensor products of representations of $U_q(\mathfrak{sl}_3(\mathbb{C}))$.

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1. INTRODUCTION

Given a complex vector space V , consider the ring $R_{a,b}(V) = \mathbb{C}[(V^*)^a \times V^b]^{\mathrm{SL}(V)}$ of polynomial functions on the space of configurations of a vectors and b covectors which are invariant under the natural action of $\mathrm{SL}(V)$. Rings of this type play a central role in representation theory, and their study dates back to Hilbert. Over the last three decades, bases of $R_{a,b}(V)$ with remarkable properties were found by Lusztig [33, 34] and independently by Kashiwara [24], and further studied by Kuperberg [27], Lusztig [35], Geiss-Schröer-Leclerc [21], Fontaine-Kremnitzer-Kuperberg [18], Gross-Hacking-Keel-Kontsevich [23]. To explicitly construct, as well as to compare, some of these bases remains a challenging problem, already open when V is 3-dimensional.

New perspectives in the study of canonical bases for $R_{a,b}(V)$ were suggested by Fomin and Pylyavskyy [14] by establishing that $R_{a,b}(V)$ has (several) structures of cluster algebras, when V is 3-dimensional. These cluster algebra structures provide a way to determine canonical bases in $R_{a,b}(V)$ by comparison with other rings which are cluster algebras and possess the notion of dual canonical bases. In particular, the set of cluster monomials forms the dual canonical basis of $R_{a,b}(V)$ when $a = 0$ and $b \leq 8$ and (conjecturally) a subset for all other values of a and b . To describe all dual canonical basis elements is an open problem already for $R_{0,9}(V)$.

This project was supported by the SNF grant P2EZP2₁48747.

The main goal of this work is to exploit earlier results on canonical bases for cluster algebras associated to Riemann surfaces in order to show that there are $SL(V)$ -invariants in $R_{a,b}(V)$ which can be described naturally by recursive operations with desirable positivity properties. The recursions we find are Chebyshev polynomials (of the first or second kind). For these polynomials we also provide a combinatorial description in the language of tensor diagrams. More precisely, we consider Chebyshev polynomials in invariants in $R_{a,b}(V)$ described by the simplest planar tensor diagrams which do not have a description as single tree diagrams. The approach we suggest is similar to the topological characterization of Chebyshev polynomials in surface cluster algebras [38] and skein algebras [42].

In the second part of this paper, we investigate the relation between the $SL(V)$ -invariants described by Chebyshev polynomials of the second kind and the dual of Lusztig's canonical basis for the invariant space of tensor products of fundamental representations of $U_q(\mathfrak{sl}_3(\mathbb{C}))$, denoted by $\text{Inv}(V)$ and defined in [33]. For this we exploit the graphical computations originating in work of [19, 25, 20] relating Lusztig's dual canonical and Kuperberg's web basis, as well as a more recent approach using the combinatorial tool of red graphs and Khovanov-Kuperberg algebras developed in [40] together with the decategorification of [36].

In particular, we consider the third and fifth Chebyshev polynomials of the second kind $U_3([W], [B(W)])$ and $U_5([W], [B(W)])$ where $[W]$ is described by the by smallest non-elliptic web with a single-cycle W and $[B(W)]$ is obtained from $[W]$. We then show that for at least one of these two polynomials, there is no dual canonical basis element with integer coefficients that specializes to $U_3([W], [B(W)])$ or $U_5([W], [B(W)])$ at $q = 1$.

Connections between Chebyshev polynomials of the second kind and Lusztig's dual canonical bases were first made explicit in [29] and in [3, 9, 28]. In these papers a deformation of the usual cluster algebra of Kronecker type was considered, in the sense of [2], arising as a subalgebra of the positive part $U_q(\mathfrak{n})$ of the universal enveloping algebra of a Kac-Moody Lie algebra of type $A_1^{(1)}$. In this situation, the specialization of Lusztig's dual canonical basis [32], at the classical limit at $q = 1$, is given by the set of all cluster monomials together with Chebyshev polynomials of the second kind in one variable parametrized by the imaginary root in the corresponding root system. Moreover, the quadratic Chebyshev polynomial of the second kind, $U_2([W], [B(W)])$, for $[W]$ and $[B(W)]$ as before, is a dual canonical basis element in $\text{Inv}(V)$, as shown in [25]. On the other side, connections between Chebyshev polynomials of the first kind and Lusztig's semi canonical basis [35] were made explicit in [29] and [21, 18].

Chebyshev polynomials and cluster algebras were first related to each other in the work of Sherman and Zelevinsky [41] where linear bases for rank 2 cluster algebras of affine types with certain nice properties were constructed. Since then Chebyshev polynomials have been linked to cluster combinatorics in several important contexts:

- Bases for surface cluster algebras and skein algebras [41, 42, 38, 6, 37].
- Canonical bases of cluster varieties related to SL_2 local systems [13].
- Theta/greedy bases for rank 2 cluster algebras [23, 8, 31, 30].
- Quiver representation theory [29, 5, 12, 11, 10].

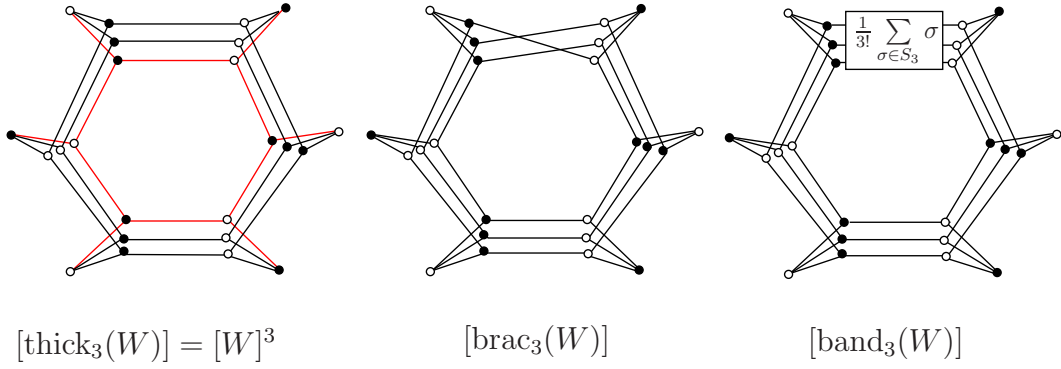


FIGURE 1. Tensor diagrams defining the invariants $[W]^3$, $[\text{brac}_3(W)]$ and $[\text{band}_3(W)]$ of the single-cycle web W of lengths six, in red in the Figure.

Our study of Chebyshev polynomials (of the first or second kind) in $R_{a,b}(V)$ should prove useful whenever one needs to recognize and distinguish different “canonical” bases of $R_{a,b}(V)$ and quantizations thereof.

1.1. Commutative case. Let V be a 3-dimensional complex vector space and let $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the dual space. Let $X = (V^*)^a \times V^b$ be the direct product of a -copies of V^* and b -copies of V for $a, b \in \mathbb{N}$. The special linear group $\text{SL}(V)$ naturally acts on X and on its coordinate ring $\mathbb{C}[X]$. Let $R_{a,b}(V) = \mathbb{C}[X]^{\text{SL}(V)}$ be the commutative associative ring of $\text{SL}(V)$ -invariant polynomials on X .

Many important invariants in $R_{a,b}(V)$ can be described graphically in terms of *tensor diagrams*. These are finite bipartite graphs with a fixed proper coloring of their vertices in two colors, black and white, and with a fixed partition of their vertices into boundary and internal vertices. Each internal vertex of a tensor diagram D is trivalent and comes with a fixed cyclic order of the edges incident to it. If the boundary of D consists of a white vertices and b black ones, one says that D has type (a, b) . The coloring of the boundary of D determines a binary cyclic word, called *signature of D* . Moreover, each invariant of $R_{a,b}(V)$ can be described uniquely by a \mathbb{C} -linear combination of invariants associated with *non-elliptic webs*. These are planar tensor diagrams such that all interior faces have at least six sides. The \mathbb{C} -linear basis spanned by invariant defined by non-elliptic webs is Kuperberg’s web basis [27].

Let $[W]$ be the invariant in $R_{a,b}(V)$ defined by an arbitrary non-elliptic web W of type (a, b) with a single internal cycle and no four cycles (in particular this excludes that W has a quadrilateral with one vertex on the boundary attached to its internal cycle).

In this paper we consider three operations applied on $[W]$. The first operation is the ordinary power, represented in [14] by superimposing k -copies of W . The invariant described in this way, can be expressed in Kuperberg’s web basis by a single non-elliptic web invariant called the *k -thickening* of W , denoted by $[\text{thick}_k(W)]$. The second operation is the *k -bracelet operation*, denoted by $[\text{brac}_k(W)]$ and represented by concatenating k copies of W in such a way that the resulting tensor diagram sits on a Möbius strip. The third operation is the *k -band operation*, denoted by $[\text{band}_k(W)]$, represented by averaging over all possible ways of joining k copies of W , placed on top of each other. For each of these operations, we bundle together k -tuples of adjacent identically colored endpoints coming from the k -copies of W . These operations are illustrated in Figure 1 for $[W]$ defined by the minimal single-cycle web W drawn in red on the left of the figure.

We give formulas for invariants associated to the bracelet and band operation of $[W]$. To state these results, we introduce the invariant $[B(W)] = \frac{1}{2}([W]^2 - [\text{brac}_2(W)])$ defined

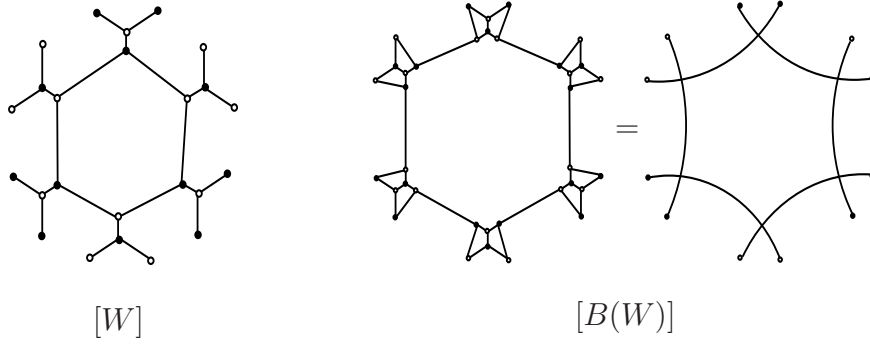


FIGURE 2. Tensor diagrams defining the invariant $[B(W)]$ associated to the invariant $[W]$ on the left.

by a tensor diagram $B(W)$ obtained from W . For every single-cycle non-elliptic web W , the tensor diagram $B(W)$ can always be described as a single non-elliptic web, or equivalently as a superimposition of some tree diagrams. In Figure 2 we illustrate the two tensor diagrams defining $[B(W)]$ corresponding to $[W]$ described by the non-elliptic web on the left hand side.

Theorem 1.1. *The k -bracelet operation of W is given by*

$$[\text{brac}_k(W)] = T_k([W], [B(W)])$$

where $T_k(x, y)$ is the rescaled Chebyshev polynomial of the first kind defined by the recurrence

$$\begin{aligned} T_0(x, y) &= 2, \\ T_1(x, y) &= x, \\ T_k(x, y) &= xT_{k-1}(x, y) - yT_{k-2}(x, y). \end{aligned}$$

Recall that the usual Chebyshev polynomials of the first kind $\text{Cheb}_k(x)$ are defined by $\text{Cheb}_k(\cos x) = \cos(kx)$, $k \in \mathbb{Z}_{\geq 0}$. The polynomials $T_k(x, y)$ are related to these by $T_k(x, y) = 2\text{Cheb}_k(\frac{x}{2\sqrt{y}})y^{k/2}$.

Theorem 1.2. *The k -band operation of W is given by*

$$[\text{band}_k(W)] = U_k([W], [B(W)])$$

where $U_k(x, y)$ is the rescaled Chebyshev polynomial of the second kind defined by the recurrence

$$\begin{aligned} U_0(x, y) &= 1, \\ U_1(x, y) &= x, \\ U_k(x, y) &= xU_{k-1}(x, y) - yU_{k-2}(x, y). \end{aligned}$$

The polynomials $U_k(x, y)$ are related to $T_k(x, y)$ by $\frac{d}{dx}T_k(x, y) = kU_{k-1}(x, y)$. Hence, $kU_{k-1}(x, y) = \frac{d}{dx}(2\text{Cheb}_k(\frac{x}{2\sqrt{y}})y^{k/2})$.

As a corollary of these results, we show that the ordinary k -power $[W]^k$ can be expressed as a positive integer linear combination of invariants associated to the band, resp. bracelet operation of $[W]$. This contrasts with expanding $[\text{brac}_k(W)]$, resp. $[\text{band}_k(W)]$, in the web basis, where one uses the polynomials $T_k([W], [B(W)])$, resp. $U_k([W], [B(W)])$, which have negative coefficients.

We next explain how Theorem 1.1 and Theorem 1.2 relate to previous work.

1.2. The cluster algebra of Kronecker type. Let $\mathcal{F} = \mathbb{Q}(x_1, x_2)$ be the field of rational functions in two commuting independent variables x_1 and x_2 with rational coefficients. Recursively define $x_j \in \mathcal{F}$ by $x_{j-1}x_{j+1} = x_j^2 + 1$. Consider the subring of \mathcal{F} generated by all $x_j, j \in \mathbb{Z}$, subject to the above relation. This subring, denoted by $\mathcal{A}_1^{(1)}$, is an example of a coefficient free cluster algebra [16]. The generators are the cluster variables and the relation is called exchange relation. The sets $\{x_j, x_{j+1}\}$, for $j \in \mathbb{Z}$, are clusters and an element of the form $x_j^k x_{j+1}^l$, for $k, l \in \mathbb{Z}_{\geq 0}$, is called a cluster monomial. This cluster algebra is of rank 2 and corresponds to the root system of affine type $A_1^{(1)}$, and therefore called of Kronecker type.

Following [41], we now recall three distinguished natural bases for $\mathcal{A}_1^{(1)}$. The topological description of these bases leads to the three operations we consider in $R_{a,b}(V)$.

Consider the distinguished variable $z = x_0x_3 - x_1x_2 \in \mathcal{A}_1^{(1)}$ expressed in $\{x_0, x_1\}$ and $\{x_2, x_3\}$. Let the Chebyshev polynomials of the first kind in this variable, $T_k(z)$, for $k \geq 1$, be defined by the same recursion as before, when setting $y = 1$.

Theorem 1.3. [41, Thm. 2.8] *The disjoint union of the set of all cluster monomials and the set $\{T_k(z) | k \geq 1\}$ is a \mathbb{Z} -linear basis of $\mathcal{A}_1^{(1)}$.*

This basis is the greedy basis in [31], which we now know coincides with the theta basis of [23], as shown in [8].

To introduce the second basis for $\mathcal{A}_1^{(1)}$ consider the family of Chebyshev polynomials of the second kind $U_k(z)$, $k \geq 1$, in the same variable z as above.

Theorem 1.4. [5] *The disjoint union of the set of all cluster monomials and the set $\{U_k(z) | k \geq 1\}$ is a \mathbb{Z} -linear basis of $\mathcal{A}_1^{(1)}$.*

This basis was called semicanonical in [5, 43] but later turned out to coincide with the specialization of Lusztig's dual canonical basis at $q = 1$, see [9, 28].

The third basis for $\mathcal{A}_1^{(1)}$ is the standard monomial basis, described in bigger generality in [1], and here given by

$$\{y_0^{a_0} y_1^{a_1} y_2^{a_2} y_3^{a_3} : a_m \in \mathbb{Z}_{\geq 0}, a_0 a_2 = a_1 a_3 = 0\}$$

where y_0, y_1, y_2, y_3 are any four consecutive cluster variables of $\mathcal{A}_1^{(1)}$. Unlike the previous two, the standard monomial basis is the one with fewer canonical properties.

Topologically, the combinatorics governing $\mathcal{A}_1^{(1)}$ can be described through (isotopy classes of) arcs inside an annulus with one marked point on each boundary component. In this approach, the variable z is parametrized by the closed loop in the interior of the annulus, denoted δ . The polynomials $T_k(z)$ can then be described by the k -bracelet operation on δ . That is, by superimposing k -copies of δ and forming a $(k-1)$ -self crossing. Similarly, the polynomials $U_k(z)$ can be described by the k -band operation on δ . That is, by averaging over all possible ways of joining parallel k -copies of δ . The last operation is the specialization of standard Jones-Wenzl idempotents, also called magic elements, at $q = 1$, see also Remark 4.22 in [42].

Much of this was later generalized to cluster algebras associated to compact oriented Riemann surfaces [15] (possibly with boundary) by [38] and to surface skein algebras in [42]. In these settings, Chebyshev polynomials of the first, resp. second, kind are taken in the variables associated to all closed non-contractible curves in the surface, and can

be described again by the bracelet, resp. band, operations on these loops.

The three operations defined in the setting of tensor diagrams in this paper, are generalizations of these operations for non-contractible curves. Moreover, our Theorem 1.1 and Theorem 1.2 can be seen as partial results towards finding bases for $R_{a,b}(V)$, generalizing Theorem 1.3, resp. Theorem 1.4. To extend the union of the disjoint sets of all cluster monomials and Chebyshev polynomials of the first, resp. second kind, in all single-cycle non-elliptic web invariants to bases of $R_{a,b}(V)$ remains an open problem.

1.3. Non-commutative case. We begin recalling the basic set up from [25]. Let $U_q(\mathfrak{sl}_3(\mathbb{C}))$ be the quantum group corresponding to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. We follow the convention of setting $v = -q^{1/2}$ and view $U_q(\mathfrak{sl}_3(\mathbb{C}))$ as an associative Hopf algebra over $\mathbb{C}(v)$. Let V° and V^\bullet be the two 3-dimensional irreducible representations of type I of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ specializing to V and V^* when $q \rightarrow 1$. Let $S = (s_1, s_2, \dots, s_n) \in \{\circ, \bullet\}^n$ be an arbitrary signature and consider the tensor product of irreducible representations $V^S = V^{s_1} \otimes V^{s_2} \dots \otimes V^{s_n}$. This tensor product will be turned again into a representation of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ and we will consider the space of invariants:

$$\text{Inv}(V^S) = \{e \in V^S : Y \cdot e = \epsilon(Y)e \text{ for all } Y \in U_q(\mathfrak{sl}_3(\mathbb{C}))\}$$

where $\epsilon : U_q(\mathfrak{sl}_3(\mathbb{C})) \rightarrow \mathbb{C}(v)$ is the counit of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ defined on the standard generators of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ by:

$$\begin{aligned} \epsilon(E_i) &= \epsilon(F_i) = 0 \\ \epsilon(K_i) &= 1. \end{aligned}$$

The vector space of all invariants is denoted by $\text{Inv}(V)$.

This space is of particular interest to us for the following reasons.

- For any signature S the dual canonical basis has been defined in $\text{Inv}(V^S)$ and hence in $\text{Inv}(V)$. This basis is dual to the canonical basis defined by Lusztig's [33] and related to the global crystal basis of Kashiwara [24] independently defined also in [32].
- At the classical limit the invariant space $\text{Inv}(V)$ specializes to the polynomial invariant ring $R_{a,b}(V)$, as we explain later.
- The quadratic Chebyshev polynomial of the second kind in the invariant defined by the smallest single-cycle web belongs to Lusztig's dual canonical basis for $\text{Inv}(V)$, see [25, Thm. 4].

More precisely, Kuperberg's web basis extends to a $\mathbb{C}(v)$ -linear basis for $\text{Inv}(V)$. The relationship between the web basis and Lusztig's dual canonical basis of $\text{Inv}(V)$ was investigated by Khovanov and Kuperberg [25]. They discovered that these two bases are in general different from each other, despite overlapping substantially and sharing several properties. To describe the discrepancy they found, in the language of this paper, consider again the non-elliptic webs defining the invariants $[\text{thick}_k(W)]$, $[\text{brac}_k(W)]$, and $[\text{band}_k(W)]$. Different invariants, which we denote by $[\text{Thick}_k(W)]$, $[\text{Brac}_k(W)]$, and $[\text{Band}_k(W)]$ can be obtained from $[\text{thick}_k(W)]$, $[\text{brac}_k(W)]$, and $[\text{band}_k(W)]$ after splitting all endpoints of the non-elliptic webs defining the invariants $[\text{thick}_k(W)]$, $[\text{brac}_k(W)]$, and $[\text{band}_k(W)]$. In $R_{a,b}(V)$ this splitting procedure represents the polarization of the corresponding invariant tensor.

Khovanov and Kuperberg show that when W is the smallest single-cycle web, the web basis element $[\text{Thick}_2(W)]$ does not lie in Lusztig's dual canonical basis; instead, the element $[\text{Band}_2(W)]$ is a dual canonical basis element, see [25]. It is then natural to

ask: do all invariants $[\text{Band}_k(W)]$, $k \in \mathbb{Z}_{\geq 0}$, belong to the specialization of Lusztig's dual canonical basis at $q = 1$?

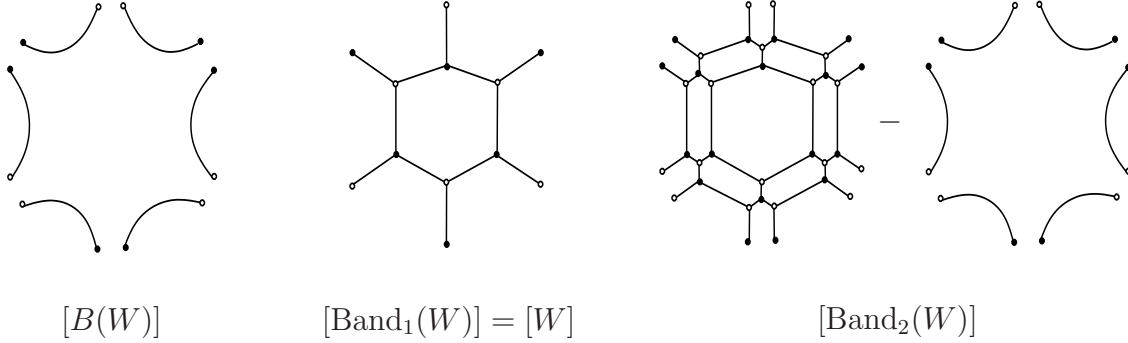


FIGURE 3. Three Lusztig's dual canonical basis elements of $\text{Inv}(V)$ expressed in Kuperberg's web basis.

We give a negative answer to this question, when the coefficients of the dual canonical basis element are assumed to be integers.

Proposition 1.5. *Let W be the minimal single-cycle non-elliptic web. There is no Lusztig's dual canonical basis element of $\text{Inv}(V)$ with integer coefficients that specializes to*

$$[\text{Band}_3(W)] \text{ or } [\text{Band}_5(W)]$$

when $q \rightarrow 1$.

The general case, in which coefficients involving q are allowed, remains open.

In the middle of Figure 3, the minimal single-cycle non-elliptic web W is shown. It can be seen that the invariant $[W] \in R_{a,b}(V)$ defined by this W plays the same role as the loop circling the annulus plays in the cluster algebra of Kronecker type discussed in Section 1.2. In fact, $[W]$ decomposes as $z_0 z_3 - z_1 z_2$ for the cluster variables in $\{z_0, z_1\}$ and $\{z_2, z_3\}$ belonging to a cluster subalgebra of $R_{a,b}(V)$ of Kronecker type, up to coefficients.

1.4. From $\text{Inv}(V)$ back to $R_{a,b}(V)$. Let $X = (V^*)^a \times V^b$. Every polynomial in $\mathbb{C}[X]$ decomposes in a unique way into a sum of multihomogeneous functions. Moreover, if a polynomial is $\text{SL}(V)$ -invariant then every multihomogeneous function is also $\text{SL}(V)$ -invariant. Hence $R_{a,b}(V)$ decomposes as follows:

$$R_{a,b}(V) = \mathbb{C}[X]^{\text{SL}(V)} = \bigoplus_{p \in \mathbb{N}^a} \bigoplus_{q \in \mathbb{N}^b} \mathbb{C}[X]_{[p,q]}^{\text{SL}(V)}.$$

To relate this space to the specialization of $\text{Inv}(V)$ at the classical limit we need to give another description of $R_{a,b}(V)$. For this let $p \in \mathbb{N}^a$ and $q \in \mathbb{N}^b$ and consider

$$(V^*)^{\otimes p} \otimes V^{\otimes q} = V^{*\otimes p_1} \otimes \dots \otimes V^{*\otimes p_a} \otimes V^{\otimes q_1} \otimes \dots \otimes V^{\otimes q_b}.$$

Then $\text{SL}(V)$ acts on V by the regular representation and on V^* by the dual representation. Hence $(V^*)^{\otimes p} \otimes V^{\otimes q}$ can be seen as a $\text{SL}(V)$ -module. Let $|p| = \sum p_i$ and $|q| = \sum q_i$. Let $\mathfrak{S}_p = \mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2} \times \dots \times \mathfrak{S}_{p_a}$ acting as a group of permutations of $\{1, \dots, |p|\}$, where the symmetric group \mathfrak{S}_{p_1} permutes the factors in position 1 up to p_1 , \mathfrak{S}_{p_2} permutes

$p_1 + 1, \dots, p_1 + p_2$, and so on. Then $\mathfrak{S}_p \times \mathfrak{S}_q$ acts on $(V^*)^{\otimes p} \otimes V^{\otimes q}$ and the actions of $\mathfrak{S}_p \times \mathfrak{S}_q$ and $\mathrm{SL}(V)$ commute. One then deduces:

$$\begin{aligned}
R_{a,b}(V) &\cong \bigoplus_{p \in \mathbb{N}^a} \bigoplus_{q \in \mathbb{N}^b} \left[((V^*)^{\otimes |p|} \otimes V^{\otimes |q|})^{\mathfrak{S}_p \times \mathfrak{S}_q} \right]^{\mathrm{SL}(V)} \\
&\cong \bigoplus_{p \in \mathbb{N}^a} \bigoplus_{q \in \mathbb{N}^b} \left[((V^*)^{\otimes |p|} \otimes V^{\otimes |q|})^{\mathrm{SL}(V)} \right]^{\mathfrak{S}_p \times \mathfrak{S}_q} \\
&\subset \bigoplus_{p \in \mathbb{N}^a} \bigoplus_{q \in \mathbb{N}^b} ((V^*)^{\otimes |p|} \otimes V^{\otimes |q|})^{\mathrm{SL}(V)} \\
&\cong \mathrm{Inv}_{\mathrm{SL}(V)}(V)
\end{aligned}$$

where the first two lines are [22, Lemma 5.4.1]. One concludes observing that $\mathrm{Inv}(V)$ specializes to $\mathrm{Inv}_{\mathrm{SL}(V)}(V)$ when $q \rightarrow 1$.

Notice that the product in $\mathrm{Inv}(V)$ is represented by disjoint union of tensor diagrams while the product in $R_{a,b}(V)$, being commutative, can be represented by superimposing the corresponding tensor diagrams (and clasping endpoints).

1.5. The quantum cluster algebra of Kronecker type. The rank 2 cluster algebra of Kronecker type $\mathcal{A}_1^{(1)}$ associated to the annulus with one marked point on each boundary component deforms to a non-commutative quantum cluster algebra, in the sense of [2]. This quantum cluster algebra is a subalgebra of the positive part of the Kac-Moody Lie algebra of type $A_1^{(1)}$. Moreover, this quantum cluster algebra is equipped with Lusztig's canonical basis for quantized enveloping algebras [32] and its dual basis. This notion of dual canonical basis is different then the Lusztig's notion for $\mathrm{Inv}(V)$, considered in this paper, but closely related, as explained in [33].

For the variable $z = x_0 x_3 - x_1 x_2 \in \mathcal{A}_1^{(1)}$ expressed in the cluster variables of $\{x_0, x_1\}$ and $\{x_2, x_3\}$ described before, the following two results are known.

Theorem 1.6. [21] *The disjoint union of the set of all cluster monomials and the set $\{T_k(z) | k \geq 1\}$ belongs to the specialization of Lusztig's semi canonical basis for $\mathcal{A}_1^{(1)}$ at $q = 1$.*

Theorem 1.7. [9] *The disjoint union of the set of all cluster monomials and the set $\{U_k(z) | k \geq 1\}$ belongs to the specialization of Lusztig's dual canonical basis for $\mathcal{A}_1^{(1)}$ at $q = 1$.*

The last result also follows from work of Lampe [28] setting trivial coefficients, see also [3, 2].

1.6. Two-dimensional case. In this setting

$$R_{a,b}(V) \cong R_{0,a+b}(V) \cong \mathbb{C}[\mathrm{Gr}_{2,a+b}(\mathbb{C})].$$

It is well known that this ring is a cluster algebra corresponding to the root system of type A_n , and that set of all cluster monomials is a linear basis for $\mathbb{C}[\mathrm{Gr}_{2,a+b}(\mathbb{C})]$, see [17]. On the other side, Kuperberg's web basis in the 2-dimensional case consists of non crossing matchings in a disc with marked points on the boundary, see [27, §.2]. If one divides the boundary points into $a + b$ intervals and avoids U turns between endpoints in the same interval, as for clasped web spaces defined in [27, §. 2.2], one can see that Kuperberg's web bases and the cluster monomial bases coincide.

Moreover, these bases are known to be the specialization of Lusztig's dual canonical bases for the invariant space in the tensor product of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ -representations, see [19].

In particular, it follows, that in the 2-dimensional setting there no Chebyshev polynomials occur in these bases.

On the other extreme, extending the web-based approach to higher dimensions remains a major outstanding problem, since the concept of the web basis has no known natural generalization for vector spaces V of dimension 4 or greater.

1.7. General plan of the paper. In Section 2 we introduce the basic notions related to the commutative polynomial ring $R_{a,b}(V)$. We define the bracelet and band operations and prove Theorem 1.1 and Theorem 1.2. Properties of bases of $R_{a,b}(V)$ including $[\text{band}_k(W)]$, resp. $[\text{brac}_k(W)]$ as basis elements will also be discussed. Some of the results in this section already appeared in my PhD thesis.

In Section 3 we pass to the non-commutative setting and describe the quantized tensor invariant space $\text{Inv}(V)$ algebraically and using tensor diagrams. The combinatorial methods of flows developed in [25] will also be revised in this section and we recall how invariants associated to non-elliptic web expand in the tensor product basis. In Section 4 we give an alternative description of the coefficients appearing in the decomposition in the tensor product basis of a collection of web invariants in $\text{Inv}(V)$, using only the boundary data of the non-elliptic web. In Section 5 we describe Lusztig's dual canonical bases for $\text{Inv}(V)$. Following [25], we recall how Lusztig's dual canonical bases relate to Kuperberg's web bases for $\text{Inv}(V)$ and give a simple operation on non-elliptic web invariants preserving the property of being dual canonical. In Section 6 we review the definition of red graphs introduced in [40] and express in the language used in this paper some key results of [40] and [36]. In Section 7 we provide some preliminary results and prove Proposition 1.5. We conclude this section formulating further possible directions of investigation in Conjecture 7.8.

1.8. Acknowledgments. I would like to thank S. Fomin and P. Pylyavskyy for introducing me to this beautiful subject during the workshop on cluster algebras at MSRI, Berkeley, in 2012. I thank both of them for the always interesting and stimulating discussions we had since then. I also thank the Department of Mathematics of the University of Michigan for its hospitality and support during my research stay here.

2. $\text{SL}(V)$ - INVARIANTS AND CHEBYSHEV POLYNOMIALS

We begin recalling some preliminary definitions and results following mainly [14].

Let $V = \mathbb{C}^3$, and $V^* = \text{Hom}_{\mathbb{C}}(V^3, \mathbb{C})$. Elements in V are *vectors*, while elements in V^* are *convector*s. Let

$$X = \underbrace{V^* \times \cdots \times V^*}_{a\text{-times}} \times \underbrace{V \times \cdots \times V}_{b\text{-times}}.$$

A *tensor T of type (a, b)* is a multilinear map $T : X \rightarrow \mathbb{C}$. Let $\text{vol} : V^3 \rightarrow \mathbb{C}$ be the *volume tensor* with evaluation $\text{vol}(v_1, v_2, v_3)$ given by the oriented volume of the parallelotope with sides v_1, v_2, v_3 . Let $\text{vol}^* : V^{*3} \rightarrow \mathbb{C}$ be the *dual volume tensor* defined by

$$\text{vol}(v_1, v_2, v_3) \text{vol}^*(u_1^*, u_2^*, u_3^*) = \det(u_j^*(v_i))_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}}.$$

Let $\text{id} : V^* \times V \rightarrow \mathbb{C}$ be the *identity tensor* corresponding to the identity operator on V . Let e_1, e_2, e_3 be the standard basis of V satisfying $\text{vol}(e_1, e_2, e_3) = 1$, then $\text{vol}^*(e_1^*, e_2^*, e_3^*) = 1$.

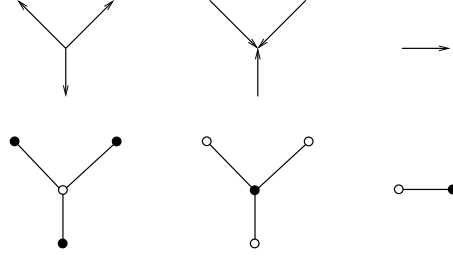


FIGURE 4. Tensor diagrams associated to vol , vol^* , and id .

The action of the special linear group $\text{SL}(V)$ on V induces a left action of $\text{SL}(V)$ on V^* given by $g \cdot u^*(v) = u^*(g^{-1} \cdot v)$ for $g \in \text{SL}(V)$, $v \in V$ and $u^* \in V^*$. The action of $\text{SL}(V)$ on X is given by $g \cdot (\underline{u}^*, \underline{v}) = (g \cdot u_1^*, \dots, g \cdot u_a^*, g \cdot v_1, \dots, g \cdot v_b)$ for $g \in \text{SL}(V)$, $(\underline{u}^*, \underline{v}) = (u_1^*, \dots, u_a^*, v_1, \dots, v_b) \in X$. The action of $\text{SL}(V)$ on the coordinate ring $\mathbb{C}[X]$ is given by

$$(g \cdot f)(\underline{u}^*, \underline{v}) = f(g^{-1} \cdot (\underline{u}^*, \underline{v}))$$

for $f \in \mathbb{C}[X]$, $g \in \text{SL}(V)$.

Definition 2.1. Let $R_{a,b}(V) = \mathbb{C}[X]^{\text{SL}(V)}$ be the ring of $\text{SL}(V)$ -invariant polynomials on X .

A description of $R_{a,b}(V)$ in coordinate notation can be found in [14, Section 2]. The tensors vol , vol^* and id are examples of $\text{SL}(V)$ -invariant polynomials on X . Also Plücker coordinates, dual Plücker coordinates and bilinear forms obtained by pairing covectors of V^* with vectors of V are $\text{SL}(V)$ -invariant polynomials on X . By the First Fundamental Theorem of $\text{SL}(V)$ -invariants $R_{a,b}(V)$ is generated by the last three types of tensors.

To reflect the order of the covariant and contravariant arguments in $R_{a,b}(V)$ one associates to elements in $R_{a,b}(V)$ a word in the alphabet $\{\circ, \bullet\}$ called *signature*. By convention the symbol \circ represents covariant arguments, while \bullet indicates the arguments of V . Then $R_\sigma(V)$ indicates the ring of $\text{SL}(V)$ -invariants of signature σ . If σ consists of a copies of \circ and b copies of \bullet we say σ has type (a, b) . It follows, that $R_\sigma(V) \cong R_{a,b}(V)$ for σ of type (a, b) .

A handy way to express elements of $R_{a,b}(V)$ is given by tensor diagrams.

Definition 2.2. A tensor diagram D is a finite bipartite graph with a fixed proper coloring of its vertices consisting of two colors, black and white, and with a fixed partition of its vertices into boundary and internal vertices. Each internal vertex of D is trivalent and comes with a fixed cyclic order of the edges incident to it. The edges of D might intersect transversally. If the boundary of D consists of a white vertices and b black ones, then D has type (a, b) .

Tensor diagrams, considered in this context, are build starting from the three building blocks associated to the tensors vol , vol^* and id . More precisely, one represent the arguments of these tensors by arrows, so that sinks correspond to vectors and sources to covectors see Figure 4. To ensure that the tensor is well defined one also specifies a cyclic ordering of the corresponding three arguments.

To obtain bigger tensors one then combines two such blocks by plugging arrowheads into arrow tails. This operation represents the contraction of the corresponding tensors.

In the sequel, arrow sinks are replaced by black colored vertices, and arrow sources by white colored vertices. Moreover, tensor diagrams are drawn inside oriented discs, in such a way that the cyclic ordering of the edges incident to each interior vertex matches

the clockwise orientation of the disk and such that the endpoints of the tensor diagram lie on the boundary.

Given an invariant $[D] \in R_{a,b}(V)$ one can substitute the same vector, or covector, into different arguments of $[D]$. This operation, called *restitution*, can be represented on tensor diagrams by boundary vertices having several incident edges. The inverse operation is called *polarization*.

Gluing boundary points however does not preserve the multilinearity of the tensor.

Given a tensor diagram D of type (a, b) an $\text{SL}(V)$ -invariant $[D]$ associated to D in $R_{a,b}(V)$ can be defined as follows: For each white vertex v of D let

$$x(v) = (x_1(v), x_2(v), x_3(v))^T$$

be the corresponding vector argument of the tensor associated to D , and for each black vertex v of D let

$$y(v) = (y_1(v), y_2(v), y_3(v))$$

be the corresponding covector argument. The invariant $[D]$ associated to D is given by

$$[D] = \sum_l \left(\prod_{v \in \text{int}(D)} \text{sign}(l(v)) \right) \left(\prod_{\substack{v \in \text{bd}(D) \\ v \text{ black}}} x(v)^{l(v)} \right) \left(\prod_{\substack{v \in \text{bd}(D) \\ v \text{ white}}} y(v)^{l(v)} \right)$$

where

- The index l runs over all proper labellings of the edges of D by the numbers 1, 2, 3. Notice that for each internal vertex v of D the labellings of the three incident edges are all distinct;
- $\text{sign}(l(v))$ is the sign of the cyclic permutation of those three labels determined by the cyclic ordering of the edges incident to v ;
- $x(v)^{l(v)}$ is the monomial $\prod_e x_{l(e)}(v)$, where e runs over all edges incident to v , similarly for $y(v)^{l(v)}$.

In [14, Example 4.2] this formula has explicitly been computed.

Tensor diagrams naturally model the additive and multiplicative structure of $R_{a,b}(V)$. Multiplication is modeled using superimposition of diagrams. Notice that when D is a union of subdiagrams D_1, D_2, \dots connected only at the boundary vertices, then $[D] = [D_1][D_2] \dots$. Addition is obtained by allowing linear combinations of tensor diagrams and extending by linearity the definition of the invariant $[D]$ associated to D .

From the First Fundamental Theorem of invariant theory it follows that any $\text{SL}(V)$ -invariant of $R_{a,b}(V)$ can be written as a linear combinations of invariants associated to tensor diagrams constructed by superimposition of the building blocks corresponding to the vol and vol^* tensors. This writing however is not unique. In Figure 5 a number of relations of invariants associated to tensor diagrams are illustrated. These relations, called *skein relations for tensor diagrams*, allow to transform a small fragment F of the diagram D into linear combinations of other diagrams $F = c_i F_i$, $c_i \in \mathbb{N}$, where the F_i are tensor diagrams of the same type as F . Whenever the tensor diagram corresponding to F defines the same invariant as $c_i F_i$, one has that $[D] = \sum_i c_i [D_i]$, where D_i indicates the tensor diagram obtained from D by replacing the fragment F with the other pieces F_i and keeping the rest of the diagram unchanged. The validity of the skein relations follow directly from the definition of tensors, see [4].

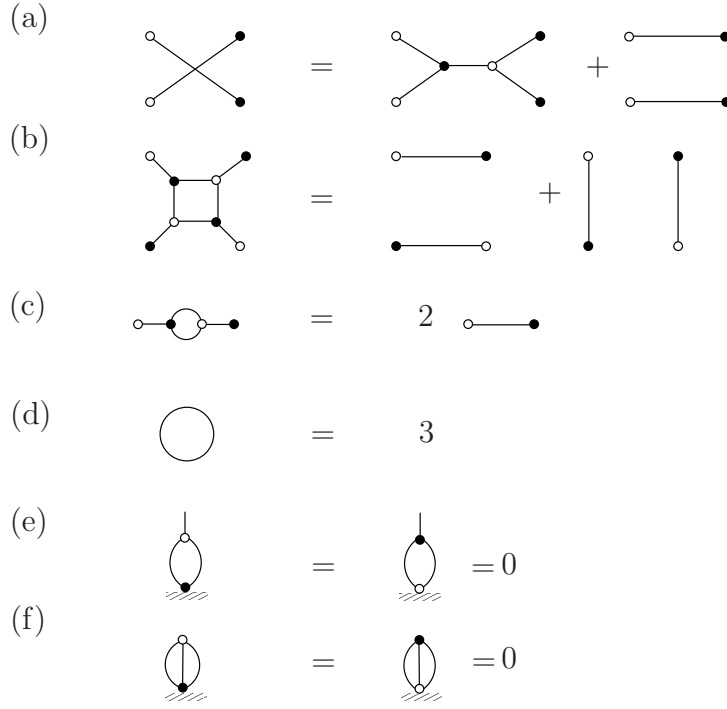


FIGURE 5

Definition 2.3. A web W is a tensor diagram embedded in an oriented disk so that its edges do not cross or touch each other, except at endpoints. Each web is considered up to isotopy of the disk that fixes its boundary.

A web is non-elliptic if it has no multiple edges and no internal cycles of length less than or equal to four.

The invariant $[W]$ associated to a non-elliptic web W is called a web invariant.

The *signature* of a web W , denoted by σ , is defined as the word in the alphabet $\{\circ, \bullet\}$ obtained from the boundary of W .

Theorem 2.4 (Kuperberg [26]). Web invariants with a fixed signature σ of type (a, b) for a linear basis in the ring of invariants $R_\sigma(V) \cong R_{a,b}(V)$.

We now turn our attention to single-cycle webs and introduce different procedures to concatenate of copies of them.

Definition 2.5. Let k be a positive integer, and let W be a non-elliptic web. The k -thickening of W is obtained as follows:

- replace each internal vertex of W by a “honeycomb” fragment H_k of the appropriated color, as shown in Figure 6(a);
- replace each edge of W by a k -tuple of edges connecting the corresponding honeycombs and / or boundary vertices.

In the sequel we denote the k -thickening of W by $\text{thick}_k(W)$.

Definition 2.6. A non-elliptic web with a single internal cycle of length at least six, is called a single-cycle web. A single-cycle web has alternating signature if adjacent boundary vertices are colored differently.

For single-cycle webs W of arbitrary signature, one can find a single-cycle web S of alternating signature inside W .

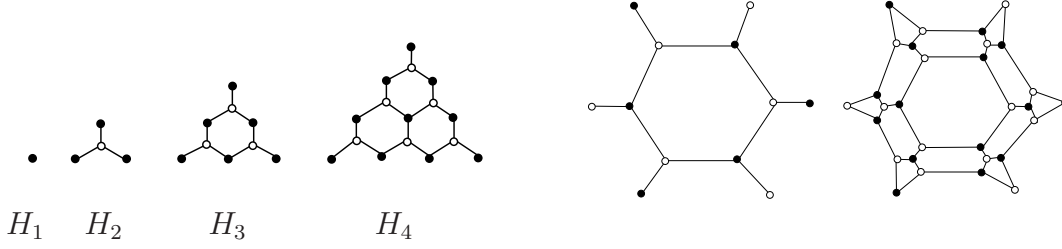


FIGURE 6. On the left honeycomb fragments are represented. On the right a single-cycle clasped web and its 2-thickening is represented.

Lemma 2.7. *Let $[W]$ be the web invariant defined by a single-cycle web W of arbitrary signature. Then, each power $[W]^k$ is a web invariant defined by $\text{thick}_k(W)$.*

Proof. If W is a single-cycle web of alternating signature the result follows from skein relation (a) and (e).

For more general single-cycle webs W , one finds a single-cycle web S contained in W of alternating signature. Let p be an internal vertex of W , which is an endpoint of S . Then there is a subtree W' of W attached to S in p . Concatenating k -copies of W one also superimposes k -copies of S , and k -copies of W' . Then $W'^k = \text{thick}_k(W')$ and S^k is given by $\text{thick}_k(S)$ in terms which attach to the honeycomb fragments H_k of $\text{thick}_k(W')$. Recursively using skein relations (b), used from the interior of the diagram towards the boundary, yields the vanishing of the term. \square

In the following definitions, let $k \in \mathbb{Z}_{\geq 0}$, and let W be a single-cycle web of arbitrary signature.

Definition 2.8. *The k -bracelet operation of W is the tensor diagram $\text{brac}_k(W)$ drawn on an oriented disk obtained by*

- *superimposing k -copies of W creating $k-1$ -self crossings*
- *gluing together the k -endpoints of the copies of W at the boundary of the disc.*

The k -bracelet operation is considered up to isotopy of the disc that fixes its boundary points.

Definition 2.9. *The k -band operation of W is the tensor diagram $\text{band}_k(W)$ drawn on an oriented disk obtained by*

- *averaging over all possible ways of concatenating the internal cycles of k copies of W*
- *gluing together the k -endpoints of the copies of W at the boundary of the disc.*

For convenience, $\text{band}_0(W) = 1$ and $\text{brac}_0(W) = 2$. In addition, $\text{band}_k(D) = \text{brac}_k(D) = \text{thick}_k(D)$ for every non-elliptic web D without any cycle (i.e. a planar tree tensor diagram).

Remark 2.10. *The bracelet and band operations can be used to detect ‘imaginary’ elements in $R_{a,b}(V)$. These are elements whose powers are not clasped web invariants, hence not expected to be cluster variables (c.f. Conjecture 9.3 in [14] and [29]).*

We now recall some basic facts about the Chebyshev recursions, which we later link to the band and bracelet operation of single-cycle tensor diagrams.

$$\begin{array}{ll}
T_0(x, y) = 2 & U_0(x, y) = 1 \\
T_1(x, y) = 1 & U_1(x, y) = x \\
T_2(x, y) = x^2 - 2y & U_2(x, y) = x^2 - y \\
T_3(x, y) = x^3 - 3xy & U_3(x, y) = x^3 - 2xy \\
T_4(x, y) = x^4 - 4x^2y + 2y^2 & U_4(x, y) = x^4 - 3x^2y + y^2 \\
T_5(x, y) = x^5 - 5x^3y + 5xy^2 & U_5(x, y) = x^5 - 4x^3y + 3xy^2 \\
\vdots & \vdots
\end{array}$$

Proposition 2.11. [38, Prop. 2.35] *For all $k \geq 1$, the monomial x^k can be written as a positive integer linear combination of the Chebyshev polynomials $T_k = T_k(x, y)$. In particular,*

$$\begin{aligned}
x^k &= T_k + \binom{k}{1}yT_{k-2} + \cdots + \binom{k}{\frac{k-1}{2}}y^{\frac{k-1}{2}}T_1, \text{ if } k \text{ is odd;} \\
x^k &= T_k + \binom{k}{1}yT_{k-2} + \cdots + \binom{k}{\frac{k-2}{2}}y^{\frac{k-2}{2}}T_2 + \binom{k}{\frac{k}{2}}y^{\frac{k}{2}}T_0, \text{ if } k \text{ is even.}
\end{aligned}$$

Similarly, for Chebyshev polynomials of the second kind we deduce the following result.

Proposition 2.12. *For all $k \geq 1$, the monomial x^k can be written as a positive integer linear combination of the Chebyshev polynomials $U_k = U_k(x, y)$. In particular, x^k is given by*

$$U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} y^{\frac{k-1}{2}} U_1$$

if k is odd; and by

$$U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k}{2} - 1} - \binom{k}{\frac{k}{2} - 2} \right\} y^{\frac{k-2}{2}} U_2 + \left\{ \binom{k}{\frac{k}{2}} - \binom{k}{\frac{k}{2} - 1} \right\} y^{\frac{k}{2}} U_0$$

if k is even.

Proof. The positivity of the linear combination follows as the differences between binomial coefficients are taken between consecutive elements in the first half of the Pascal triangle.

For the rest of the proof we proceed by cases, as for Chebyshev polynomials for the first kind, see [38, Prop. 2.35].

If k is odd, using the induction assumption, we have

$$x \left(U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} y^{\frac{k-1}{2}} U_1 \right).$$

Since $xU_k = U_{k+1} + yU_{k-1}$, we rewrite x^{k+1} as

$$\begin{aligned}
& x \left(U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} y^{\frac{k-1}{2}} U_1 \right) \\
&= U_{k+1} + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-1} + \cdots + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} y^{\frac{k-1}{2}} U_2 \\
&+ y \left(U_{k-1} + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y^2 U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k-1}{2} - 1} - \binom{k}{\frac{k-1}{2} - 2} \right\} y^{\frac{k-1}{2}} U_2 \right. \\
&\left. + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} y^{\frac{k+1}{2}} U_0 \right).
\end{aligned}$$

For the coefficient of U_0 we observe that

$$\begin{aligned}
\binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} &= \frac{2}{k+1} \binom{k+1}{\frac{k-1}{2}} \\
&= \frac{2}{k+1} \binom{k+1}{\frac{k+1}{2} + 1} \\
&= \frac{1}{\frac{k+1}{2} + 1} \binom{k+1}{\frac{k+1}{2}} \\
&= \binom{k+1}{\frac{k+1}{2}} - \binom{k+1}{\frac{k+1}{2} - 1}.
\end{aligned}$$

We conclude using Pascal's rule.

If k is even,

$$x^{k+1} = x \left(U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k}{2}} - \binom{k}{\frac{k}{2} - 1} \right\} y^{\frac{k}{2}} U_0 \right).$$

This is the same as

$$\begin{aligned}
& U_{k+1} + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y U_{k-1} + \cdots + \left\{ \binom{k}{\frac{k}{2}} - \binom{k}{\frac{k}{2} - 1} \right\} y^{\frac{k}{2}} x \\
&+ y U_{k-1} + \left\{ \binom{k}{1} - \binom{k}{0} \right\} y^2 U_{k-3} + \cdots + \left\{ \binom{k}{\frac{k}{2} - 1} - \binom{k}{\frac{k}{2} - 2} \right\} y^{\frac{k}{2}} U_1.
\end{aligned}$$

Observe that $x = U_1$, and conclude using Pascal's rule. \square

As before, let W be a single-cycle web of arbitrary signature. We will now construct a non-elliptic web $B(W)$, associated to W . The invariant described by $B(W)$ will later play the role of the coefficient in the Chebyshev recursions.

Definition 2.13. *Let S be the single-cycle clasped web contained in W of alternating signature. Then $B(W)$ is obtained from $\text{thick}_2(W)$ by removing $\text{thick}_2(S)$ in $\text{thick}_2(W)$ and joining the vertices formerly serving as endpoints of $\text{thick}_2(S)$.*

As an example, the invariant $[B(W)]$ obtained from $\text{thick}_2(W)$ as described above, for the single-cycle hexagonal non-elliptic web having a tripod attached at each end the, is illustrated in Figure 2.

Lemma 2.14. *Let k be an integer. Let W be a single-cycle web of arbitrary signature. Assume that W has no quadrilateral with one vertex on the boundary attached to the internal single cycle. Then in the k -thickening of W the following local identities hold:*

(a)

(b)

(c)

(d)

Proof. Part (a) follows by solving the first crossing with the skein relation.

Part (b) follows iteratively using skein relations on the first crossing.

Part (c) follows by the multiplicative structure of tensor diagrams and skein relations. More precisely, since no quadrilateral with one vertex on the boundary attached to the internal single cycle, iteratively solving squares generates both new squares around the cycle as well as terms eventually vanishing (by skein relation (e)). The final collision of squares, around the internal cycle, produces the factor -2 .

By similar reasoning as in Part (c) we deduce the first equality in Part (d) observing that solving the upper square in the first summand of the expansion produces the first term on the right hand side of the equality. Solving the upper squares in the remaining summands, and summing the resulting terms, produces the second term, by the identity in Part (b). The second equality follows from Part (a)

□

Remark 2.15. *If W has a quadrilateral with one vertex on the boundary attached to the internal single cycle then part (c) of the previous lemma leads to the vanishing of the corresponding invariant in $R_{a,b}(V)$.*

Proof of Thm. 1.1. The initial condition $T_1([W], [B(W)]) = [W]$ is clear and a simple computation gives

$$T_2([W], [B(W)]) = [W]^2 - 2[B(W)].$$

The recurrence then follows from Lemma 2.14 part (a) followed by part (b), (c) and (d). \square

Keeping the assumptions of Thm. 1.1 the next result can be deduced by Proposition 2.11.

Corollary 2.16. *The monomial $[W]^k$ can be written by a positive linear combination of the Chebyshev polynomials of the first kind $T_k = T_k([W], [B(W)])$. In particular,*

$$[W]^k = T_k + \binom{k}{1}[B(W)]T_{k-2} + \cdots + \binom{k}{(k-1)/2}[B(W)]^{(k-1)/2}T_1, \text{ if } k \text{ is odd};$$

$$[W]^k = T_k + \binom{k}{1}[B(W)]T_{k-2} + \cdots + \binom{k}{(k-2)/2}[B(W)]^{(k-2)/2}T_2 + \binom{k}{k/2}[B(W)]^{k/2}$$

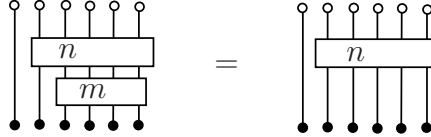
if k is even. \square

Our coefficient $[B(W)]$ in the expression for Chebyshev polynomials of the first kind is given by a product of coefficient variables for the given cluster algebra structure. To see this it suffices to remove the internal trivalent vertices of $B(W)$, using skein relations, and compare with Definition 6.1. in [14].

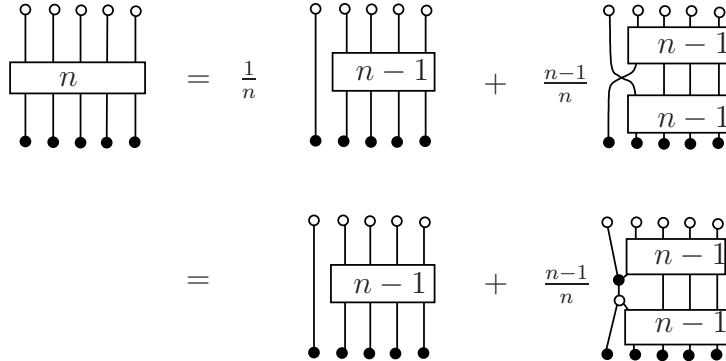
It follows that $[B(W)]$ can be thought of as a generalization of the coefficient described in [38] associated to closed contractible curves in cluster algebras associated to 2-dimensional Riemann surfaces with non-empty boundary.

Lemma 2.17. *Let k be an integer. Let W be a single-cycle web of arbitrary signature. In the thickening of W the following local identities hold:*

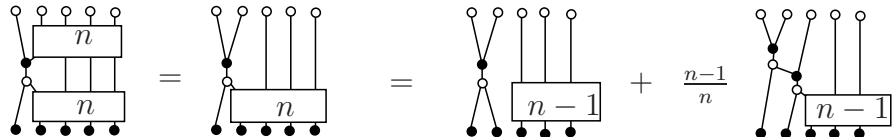
(a)



(b)



(c)



Proof. The proof is a small adaptation of the surface case, treated in [42, Lemma 4.6]. The second equality in claim (b) follows from the skein relations in the thickening of a single cycle web and claim (a).

The last claim follows from claim (b) and (a) and after having moving one box around the cycle using skein relations. \square

Proof of Thm. 1.2. Proceed by induction. For small n the claim can be checked directly. For $n > 2$, the claim can be deduce from Lemma 2.17 part (b), followed by part (c) and solving the squares in the two summands of part (c) following the reasoning of Lemma 2.14 parts (c) and (d). \square

Keeping the assumptions of Theorem 1.2 we deduce the next result with Proposition 2.12.

Corollary 2.18. *The monomial $[W]^k$ can be written by a positive linear combination of the Chebyshev polynomials of the second kind $U_k = U_k([W], [B(W)])$. In particular, $[W]^k$ is equal to*

$$U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} [B(W)] U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k-1}{2}} - \binom{k}{\frac{k-1}{2} - 1} \right\} [B(W)]^{\frac{k-1}{2}} U_1$$

if k is odd. Respectively, given by

$$U_k + \left\{ \binom{k}{1} - \binom{k}{0} \right\} [B(W)] U_{k-2} + \cdots + \left\{ \binom{k}{\frac{k}{2} - 1} - \binom{k}{\frac{k}{2} - 2} \right\} [B(W)]^{\frac{k-2}{2}} U_2 + \left\{ \binom{k}{\frac{k}{2}} - \binom{k}{\frac{k}{2} - 1} \right\} [B(W)]^{\frac{k}{2}} U_0$$

if k is even. \square

Remark 2.19. *From the previous results we see that the expansion of $[W]^k$ in the web basis uses Chebyshev polynomials $T_k([W], [B(W)])$, resp. $U_k([W], [B(W)])$, which have negative coefficients. However, $[W]^k$ expands positively in bases containing Chebyshev polynomials $T_k([W], [B(W)])$, resp. $U_k([W], [B(W)])$, see Corollary 2.16 and Corollary 2.18. This matches with the expectations coming from the surface case, as described in remark 4.8 in [38].*

Remark 2.20. *A computation shows that the previous results also hold for more general web invariants defined by non-elliptic webs which arborize to tensor diagrams with a single internal cycle. An example of such a non-elliptic web together with it's arborized form can be found at the bottom of Figure 31 in [14].*

3. THE $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -INVARIANT SPACE $\text{Inv}(V)$

In this section we follow [25] closely.

Let $\mathbb{C}(q^{1/2})$ be the field of complex-valued rational functions in the indeterminate $q^{1/2}$. Let $U_q(\mathfrak{sl}_3(\mathbb{C}))$ be the quantum group corresponding to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Then $U_q(\mathfrak{sl}_3(\mathbb{C}))$ is an associative algebra over $\mathbb{C}(q^{1/2})$, with generators E_i, F_i, K_i and K_i^{-1} ,

$i = 1, 2$, satisfying certain relations, see [25, §. 2]. Denote the quantum integer by

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}},$$

so for example $[1] = 1$, $[2] = q^{1/2} + q^{-1/2}$, $[3] = q + 1 + q$. In the following, set $v = -q^{1/2}$.

A Hopf algebra structure on $U_q(\mathfrak{sl}_3(\mathbb{C}))$ over $\mathbb{C}(v)$ can be defined setting up the co-product $\overline{\Delta}$ as follows:

$$\begin{aligned}\overline{\Delta}(K_i^\pm) &= K_i^\pm \otimes K_i^\pm \\ \overline{\Delta}(E_i) &= E_i \otimes 1 + K_i^{-1} \otimes E_i \\ \overline{\Delta}(F_i) &= F_i \otimes K_i + 1 \otimes F_i.\end{aligned}$$

A representation of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ over $\mathbb{C}(v)$ is a left $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -module. A vector e in an arbitrary $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -module V is an *invariant vector*, if $Xe = \epsilon(X)e$, for all $X \in U_q(\mathfrak{sl}_3(\mathbb{C}))$ and where $\epsilon : U_q(\mathfrak{sl}_3(\mathbb{C})) \rightarrow \mathbb{C}(v)$ is the counit of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ defined on the generators by:

$$\begin{aligned}\epsilon(E_i) &= \epsilon(F_i) = 0 \\ \epsilon(K_i) &= 1.\end{aligned}$$

The $\mathbb{C}(v)$ -vector space of all invariant vectors of V will be denoted by $\text{Inv}(V)$. Notice that $\text{Inv}(V)$ is both a subspace and a quotient space of V .

The counit ϵ of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ can also be used to turn the tensor product of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representations into $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representations. Similarly, the antipode of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ turns the dual space $V^* = \text{Hom}_{\mathbb{C}(v)}(V, \mathbb{C}(v))$ of a $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representation into a $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representation.

For two arbitrary $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representation V_1 and V_2 consider the following isomorphism:

$$V_1^* \otimes V_2 \cong \text{Hom}_{\mathbb{C}(v)}(V_1, V_2)$$

of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representations. It follows that the invariant space of a tensor product of representations can equivalently be described as

$$\text{Inv}(V_1^* \otimes V_2) \cong \text{Hom}_{U_q(\mathfrak{sl}_3(\mathbb{C}))}(V_1, V_2).$$

Let $V^\circ = V_q(\lambda)$ be the 3-dimensional irreducible $U_q(\mathfrak{sl}_3(\mathbb{C}))$ -representation parametrized by the highest weight λ . Let $V^\bullet = (V^\circ)^*$ be its dual representation. The action of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ on V° and V^\bullet is described in [25]. Each object in the category of finite dimensional representations of $U_q(\mathfrak{sl}_3(\mathbb{C}))$ can be written as a direct sum of copies of V° and V^\bullet , see [7, Theorem 10.1.7].

Let S be again a *signature*, that is a cyclic word $S = (s_1, s_2, \dots, s_n) \in \{\circ, \bullet\}^n$. Let J be a *state string*, that is a (non-cyclic) word $J = (j_1, j_2, \dots, j_n) \in \{-1, 0, 1\}^n$. Let $e_{-1}^\circ, e_0^\circ, e_1^\circ$, resp. $e_{-1}^\bullet, e_0^\bullet, e_1^\bullet$, be a basis for V° , resp. V^\bullet . The tensor product basis of

$$V^S = V^{s_1} \otimes V^{s_2} \otimes \dots \otimes V^{s_n}$$

has as basis vectors the *simple tensors* given by

$$e_J^S = e_{j_1}^{s_1} \otimes e_{j_2}^{s_2} \otimes \dots \otimes e_{j_n}^{s_n},$$

for all J and a fixed S as above. At this stage it might be useful to recall that there is no natural action of the symmetric group on the invariant space of n tensor factors with $n > 2$.

By the fundamental theorem of invariant theory, and for every signature S , all invariants in $\text{Inv}(V^S)$ can be obtained from the following *fundamental invariants*:

$$[U^{\circ\bullet}] : \mathbb{C}(v) \rightarrow V^\circ \otimes V^\bullet$$

$$1 \mapsto e_1^\circ \otimes e_{-1}^\bullet + v^{-1} e_0^\circ \otimes e_0^\bullet + v^{-2} e_{-1}^\circ \otimes e_1^\bullet$$

$$[U^{\bullet\circ}] : \mathbb{C}(v) \rightarrow V^\bullet \otimes V^\circ$$

$$1 \mapsto e_1^\bullet \otimes e_{-1}^\circ + v^{-1} e_0^\bullet \otimes e_0^\circ + v^{-2} e_{-1}^\bullet \otimes e_1^\circ$$

$$[T^{\circ\circ\circ}] : \mathbb{C}(v) \rightarrow V^\circ \otimes V^\circ \otimes V^\circ$$

$$1 \mapsto e_1^\circ \otimes e_0^\circ \otimes e_{-1}^\circ + v^{-1} e_0^\circ \otimes e_1^\circ \otimes e_{-1}^\circ$$

$$+ v^{-1} e_1^\circ \otimes e_{-1}^\circ \otimes e_0^\circ + v^{-2} e_0^\circ \otimes e_{-1}^\circ \otimes e_1^\circ$$

$$+ v^{-2} e_{-1}^\circ \otimes e_1^\circ \otimes e_0^\circ + v^{-3} e_{-1}^\circ \otimes e_0^\circ \otimes e_1^\circ$$

$$[T^{\bullet\bullet\bullet}] : \mathbb{C}(v) \rightarrow V^\bullet \otimes V^\bullet \otimes V^\bullet$$

$$1 \mapsto e_1^\bullet \otimes e_0^\bullet \otimes e_{-1}^\bullet + v^{-1} e_0^\bullet \otimes e_1^\bullet \otimes e_{-1}^\bullet$$

$$+ v^{-1} e_1^\bullet \otimes e_{-1}^\bullet \otimes e_0^\bullet + v^{-2} e_0^\bullet \otimes e_{-1}^\bullet \otimes e_1^\bullet$$

$$+ v^{-2} e_{-1}^\bullet \otimes e_1^\bullet \otimes e_0^\bullet + v^{-3} e_{-1}^\bullet \otimes e_0^\bullet \otimes e_1^\bullet$$

with the operations of tensor product and contraction:

$$[\sigma^{\bullet\circ}] : V^\circ \otimes V^\bullet \rightarrow \mathbb{C}(v), \quad e_{-1}^\circ \otimes e_1^\bullet \mapsto 1 \quad e_0^\circ \otimes e_0^\bullet \mapsto v \quad e_1^\circ \otimes e_{-1}^\bullet \mapsto v^2;$$

$$[\sigma^{\circ\bullet}] : V^\bullet \otimes V^\circ \rightarrow \mathbb{C}(v), \quad e_{-1}^\bullet \otimes e_1^\circ \mapsto 1 \quad e_0^\bullet \otimes e_0^\circ \mapsto v \quad e_1^\bullet \otimes e_{-1}^\circ \mapsto v^2,$$

and where all other basis elements are sent to zero.

Pretty much as in the commutative case, invariants in $\text{Inv}(V^S)$ can be described in the diagrammatic language of tensor diagrams, see [39, Thm. 5.1]. For this, one first associates planar graphs with oriented edges to the fundamental invariants described above, see Figure 7. The orientation of the edges is such that sinks correspond to vectors and sources to covectors arguments.

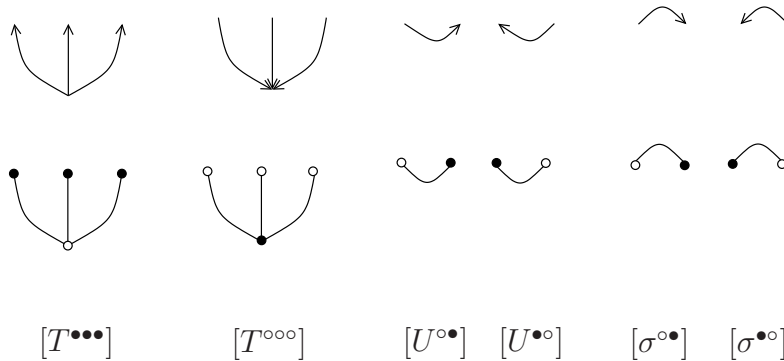


FIGURE 7

Bigger graphs can be obtained in two ways: by the disjoint union of diagrams; or by combining T blocks with σ blocks, resp. σ 's with T 's, by plugging arrowheads into

arrow tails. Similarly, one assembles σ 's and U 's, resp. U with σ 's. The resulting graph simplifies according to the following identities:

$$\begin{aligned} [(I_{V^\bullet} \otimes \sigma^{\circ\bullet}) \circ (U^{\bullet\circ} \otimes I_{V^\bullet})] &= [I_{V^\bullet}] = (\sigma^{\bullet\circ} \otimes I_{V^\bullet}) \circ (I_{V^\bullet} \otimes U^{\circ\bullet}) \\ [(I_{V^\circ} \otimes \sigma^{\bullet\circ}) \circ (U^{\circ\bullet} \otimes I_{V^\circ})] &= [I_{V^\circ}] = (\sigma^{\circ\bullet} \otimes I_{V^\circ}) \circ (I_{V^\circ} \otimes U^{\bullet\circ}). \end{aligned}$$

The disjoint union operation represents the tensor product of the associated invariants, while the “plugging in” operation expresses the contraction of the corresponding simple tensors. Iterating these operations one obtains bigger tensor diagrams, which are invariant under isotopy, and represent well defined invariants.

In the sequel, we replace arrow sinks by black colored vertices, and arrow sources by white colored vertices. In the following, sometimes the order of the colored external vertices of these tensor diagrams will be important and an initial vertex will then be specified.

Definition 3.1. *In the non-commutative setting we denote by p^+ (resp. p^-) the segments in the tensor diagram D corresponding to σ (resp. U) tensors. We let $\text{tgn}(D)$ be the set of p^+ and p^- pieces of D .*

In the following, we consider the $\mathbb{C}(v)$ -linear space spanned by all tensor diagrams with fixed boundary S quotiented by the non-commutative *skein relations* described in Figure 8. Also in this context there are relations involving positive and negative crossings of edges, see [27, §.4.]. Notice that in this context multiple edges meeting the boundary are not allowed.

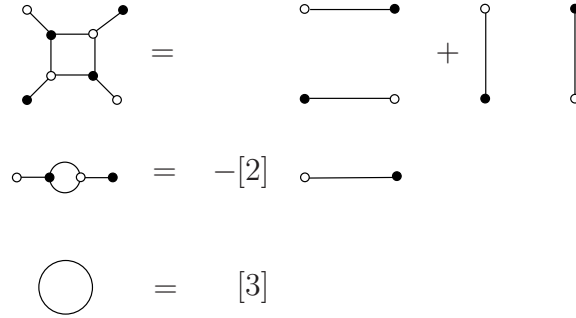


FIGURE 8

Theorem 3.2 ([27]). *Web invariants associated to non-elliptic webs with fixed signature S form a $\mathbb{C}(v)$ -linear basis for $\text{Inv}(V^S)$.*

3.1. Combinatorial description of the tensor product basis for $\text{Inv}(V)$. In the given bases of V° and V^\bullet , the invariant $[W] \in \text{Inv}(V^S)$ associated to a non-elliptic web W can be determine in various equivalent ways. One way is given by contracting the coordinate expression of the fundamental invariants, a second descriptions arises using flow lines on the associated webs. We review these descriptions below, more details can be found in [25].

Let D be any tensor diagram obtained from T, U, σ 's and with signature S .

A *state* is a labeling of the edges incident to trivalent vertices of D by the numbers $-1, 0, 1$. The labels associated with edges incident to trivalent vertices are required to be both: all distinct, and such that the sum of the labels associated to edges incident to each trivalent vertex of D and incident to the points p^+ and p^- of D is zero. A unique monomial in the coordinate expression of $[T], [U]$ or $[\sigma]$ is then associated to each choice

of state at each trivalent vertex and point p^+ , resp. p^- . The coefficient of this monomial is called the *weight of the state*. The *boundary state* is a labeling of the boundary edges of D by the numbers $-1, 0, 1$. Each boundary state extends to (possibly) several states of D . The weight of each such state is given by the product of the weights assigned locally to each trivalent vertex and p^+ and p^- . The *weight of the boundary state*, also called state sum, is the sum of weights over all states of D , arising as extensions of the given boundary state. By construction, the weights of the various boundary states of D are the coefficients of the invariant $[D]$ expanded in the tensor product basis.

This reasoning leads to the following expression of $[D] \in \text{Inv}(V^S)$:

$$(1) \quad [D] = \sum_J \left(\sum_{\ell} \left(\prod_{\substack{x \in \text{int}(D) \\ p^{\pm} \in \text{tgn}(D)}} v^{\delta_{i,j}(p^{\pm}) - \sigma(\ell(x))} \right) \right) e_J^S$$

where

- J runs over all boundary states of D ;
- ℓ runs over all states of D extending the boundary state J ;
- $\sigma(\ell(x))$ denotes the number of inversions of the state at each internal trivalent vertex x of D ;
- $\delta_{i,j}(p^{\pm})$, for $i, j \in \{-1, 0, 1\}$, is given by

$$\delta_{i,j}(p^+) = \begin{cases} 0 & \text{if } i = -1, j = 1 \\ 1 & \text{if } i = j = 0 \\ 2 & \text{if } i = 1, j = -1 \end{cases} \quad \text{resp.} \quad \delta_{i,j}(p^-) = \begin{cases} 0 & \text{if } i = 1, j = -1 \\ -1 & \text{if } i = j = 0 \\ -2 & \text{if } i = -1, j = 1 \end{cases}$$

for p^+ , resp. p^- , tangent points of D corresponding to σ , resp. U .

- e_J^S is the simple tensor determined by the signature S and state string J .

Example 3.3. Consider the tensor diagram $T^{\circ\circ\circ}$, then $S = (\circ, \circ, \circ)$ and here $J = \ell$. In addition, checking inversions of possible states of T we deduce that

- $(1, 0, -1)$ has no inversion as $1 > 0 > -1$;
- $(0, 1, -1), (1, -1, 0)$ have 1 inversion;
- $(0, -1, 1), (-1, 1, 0)$ have 2 inversion;
- $(-1, 0, 1)$ has 3 inversion.

Then equation (1) becomes

$$\begin{aligned} [T^{\circ\circ\circ}] &= \sum_J v^{-\sigma(J)} e_J^S \\ &= v^0 e_1^{\circ} \otimes e_0^{\circ} \otimes e_{-1}^{\circ} + v^{-1} e_0^{\circ} \otimes e_1^{\circ} \otimes e_{-1}^{\circ} + v^{-1} e_1^{\circ} \otimes e_{-1}^{\circ} \otimes e_0^{\circ} \\ &\quad + v^{-2} e_0^{\circ} \otimes e_{-1}^{\circ} \otimes e_1^{\circ} + v^{-2} e_{-1}^{\circ} \otimes e_1^{\circ} \otimes e_0^{\circ} + v^{-3} e_{-1}^{\circ} \otimes e_0^{\circ} \otimes e_1^{\circ} \end{aligned}$$

which agrees with the description of the fundamental invariant $[T^{\circ\circ\circ}]$ given earlier.

Remark 3.4. Notice that, since $v = -q^{1/2}$, $v^{\delta_{i,j}(p^{\pm}) - \sigma(\ell(x))} = (-q^{1/2})^{\delta_{i,j}(p^{\pm}) - \sigma(\ell(x))}$. As contractions have weight 1 when $q \rightarrow 1$, $\delta_{i,j}(p^{\pm}) = 0$ for all $p^{\pm} \in \text{tgn}(D)$ when $q \rightarrow 1$. This shows that equation (1) specializes to the determinantal formula (4.1) in [14] when $q \rightarrow 1$.

Let W be a non-elliptic web and $[W] \in \text{Inv}(V^S)$ the corresponding web invariant. Our next goal is to describe weights of states using flows and flow lines on W . A *flow line* is

an oriented closed or open path in W connecting boundary vertices. A *flow* F on W is a collection of open and closed flow lines, visiting exactly two out of three edges incident to each trivalent vertex of W . Notice that the orientation of the various flow lines is independent of the signature of W . At the boundary of W the orientation of flow lines in F determines a state string $J = (j_1, j_2, \dots, j_n) \in \{1, 0, -1\}^n$ of W according to the following convention: if the flow line points away the corresponding entry of J is 1; if the orientations is towards to the boundary vertex the state is -1 ; if the boundary segment has no flow line, the state is 0. The state string J , obtained in this way, is the boundary state of F . To compute the weight of the boundary state, it is handy to consider, together with the invariant $[U]$ and $[\sigma]$, the invariants $[Y]$ and $[\lambda]$ given by:

$$\begin{aligned} [Y_{\bullet}^{\circ\circ}] : V^{\bullet} &\rightarrow V^{\circ} \otimes V^{\circ} \\ [Y_{\bullet}^{\circ\circ}] : V^{\circ} &\rightarrow V^{\bullet} \otimes V^{\bullet} \\ [\lambda_{\bullet\bullet}^{\circ}] : V^{\bullet} \otimes V^{\bullet} &\rightarrow V^{\circ} \\ [\lambda_{\circ\circ}^{\bullet}] : V^{\circ} \otimes V^{\circ} &\rightarrow V^{\bullet} \end{aligned}$$

where $[Y_{\bullet}^{\circ\circ}]$ is defined by

$$\begin{aligned} e_1^{\bullet} &\mapsto e_1^{\circ} \otimes e_0^{\circ} + v^{-1} e_0^{\circ} \otimes e_1^{\circ} \\ e_0^{\bullet} &\mapsto e_1^{\circ} \otimes e_{-1}^{\circ} + v^{-1} e_{-1}^{\circ} \otimes e_1^{\circ} \\ e_{-1}^{\bullet} &\mapsto e_0^{\circ} \otimes e_{-1}^{\circ} + v^{-1} e_{-1}^{\circ} \otimes e_0^{\circ} \end{aligned}$$

and $[\lambda_{\bullet\bullet}^{\circ}]$ is defined by

$$\begin{aligned} e_1^{\bullet} \otimes e_0^{\bullet} &\mapsto v e_1^{\circ} & e_0^{\bullet} \otimes e_1^{\bullet} &\mapsto e_1^{\circ} \\ e_1^{\bullet} \otimes e_{-1}^{\bullet} &\mapsto v e_0^{\circ} & e_{-1}^{\bullet} \otimes e_1^{\bullet} &\mapsto e_0^{\circ} \\ e_0^{\bullet} \otimes e_{-1}^{\bullet} &\mapsto v e_{-1}^{\circ} & e_{-1}^{\bullet} \otimes e_0^{\bullet} &\mapsto e_{-1}^{\circ}. \end{aligned}$$

and zero otherwise. Similarly, one defines $[Y_{\circ}^{\bullet\bullet}]$ and $\lambda_{\circ\circ}^{\bullet}$.

These invariants are obtained from the fundamental invariants $[T^{\circ\circ\circ}]$, $[T^{\bullet\bullet\bullet}]$ by composition with $[\sigma]$. For example $[Y_{\bullet}^{\circ\circ}] = [(I_{V^{\circ\circ}} \otimes \sigma_{\bullet\bullet}) \circ (T^{\circ\circ\circ} \otimes I_{V^{\bullet}})]$.

The coefficients of $[Y]$, $[\lambda]$, $[\sigma]$, $[U]$ have been described in [25] using weights. A *local weight* is an element of $\{v^{-1}, 1, v\}$ assigned locally to a flow line on the corresponding Y , λ , U and σ web. The coloring of the boundary vertices is immaterial.

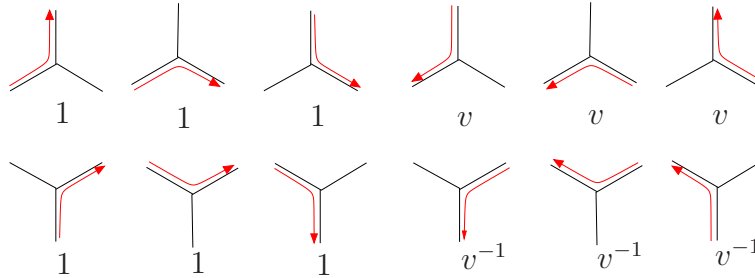


FIGURE 9

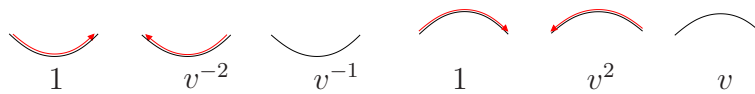


FIGURE 10

Let $F = f_1 \cup f_2 \cup \dots \cup f_n$ be a flow on W . The product over all local weights of a flow line f_i of F is the *total weight* of f_i . The *weight* of F is defined as the product of total weights of all flow lines $f_i \in F$. The *weight of a boundary state* of W , is defined as the sum of weights of all flows of W having the same boundary state.

In this way, weights of boundary states in W are the non-zero coefficients of the invariant $[W]$ expanded in the tensor product basis. On non-elliptic webs, flow lines have overall weight 1 or a power of v . Moreover, since weights don't cancel in state sums, it follows that any state sum takes values in $\mathbb{Z}_{\geq 0}[v^{-1}, v]$.

Example 3.5. Consider the invariant $[W] = (I_{V^\bullet} \otimes I_{V^\bullet} \otimes \sigma_{\bullet\bullet} \otimes I_{V^\circ} \otimes I_{V^\circ}) \circ (T^{\bullet\bullet\bullet} \otimes T^{\circ\circ\circ})$. In the tensor product basis $[W]$ decomposes into twelve monomials. On the other side, there are twelve boundary states on the non-elliptic web obtained by contracting two T 's. It is not hard to see that the weights of the various boundary states and the coefficients of the monomials in the tensor product expansion coincide.

To compute the coefficients of $[W]$ expanded in the tensor product basis, using flow lines, one needs to determine the Y, λ, U and σ pieces of W . For non-elliptic webs this can be done with the minimal cut path and growth algorithms [25, §.5], as these algorithms provide a way to build W out of smaller pieces. To describe these algorithms it is necessary to assume that W is a non-elliptic web with endpoints ordered linearly rather than cyclically. The *minimal cut path algorithm* takes as input a non-elliptic web W and determines both a signature S (which can be read off from the boundary of W) and a unique state string of W , called a *dominant lattice path* and denoted $J(W)$.

Next, consider a web of shape H obtained concatenating a Y and a λ . The *growth algorithm* takes a sign and state string (S, J) (displaced on a line) as input, and produces a web by inductively concatenating Y, H and U pieces. More precisely, the algorithm starts with a web, embedded into a half-plane, consisting of parallel strands. The boundary of the strands are colored according to S and J . One proceeds concatenating Y, H and U pieces to substrings of (S, J) , following certain simple rules described in [25, §.5]. The order in which one concatenates the new pieces is irrelevant, see Lemma 1 in [25]. Moreover, the web produced by the growth algorithm is always non-elliptic, see Lemma 2 in [25].

Proposition 3.6. [25, Prop.1] *The minimal cut path and the growth algorithm are inverses.*

It follows that, for a state string $S = (s_1, s_2, \dots, s_n)$ non-elliptic webs of $\text{Inv}(V^S)$ can be indexed by their dominant lattice path.

We will use the notation $J' < J$ for the *lexicographic order* on the set of state strings (of fixed length) in the alphabet $\{1 > 0 > -1\}$.

Theorem 3.7. [25, Thm.2] *The invariant $[W] \in \text{Inv}(V^S)$ associated to the non-elliptic web W expands as*

$$[W] = e_{J(W)}^S + \sum_{J' < J(W)} c(S, J(W), J') e_{J'}^S$$

for some coefficients $c(S, J(W), J') \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

4. ALTERNATIVE EVALUATION OF MONOMIAL WEBS IN $\text{Inv}(V)$

The aim of this section is to give an alternative combinatorial description of the decomposition of certain web invariants in the tensor product basis.

Throughout, let $W_{\lambda,Y}$ be a non-elliptic web which can be drawn inside a disk so that

- each internal (trivalent) vertex is a λ or a Y , and no adjacent vertices are of the same type;
- each edge between internal vertices contains no points with horizontal tangent.

Remark 4.1. *We conjecture that any non-elliptic web satisfies this assumption.*

Let \overline{W} be the non-elliptic web obtained from $W_{\lambda,Y}$ with all U , resp. σ , shaped boundary edges removed. Label the i -th boundary point of \overline{W} with $p_i \in \{N, NE, SE, S, SW, NW\}$, so that p_i records the position of the i -th boundary edge of \overline{W} . We call the string $P = (p_1, p_2, \dots, p_n)$ obtained in this way the *position string* of \overline{W} .

For $1 \leq k \leq n$ denote by (p_k, j_k) the k boundary edge of \overline{W} with state j_k and in position p_k . To every irreducible rotation of the edge (p_k, j_k) around $\frac{2\pi}{6}$ we assign an element of $\{v^{-1}, 1, v\}$ according to the chart of Figure 11.

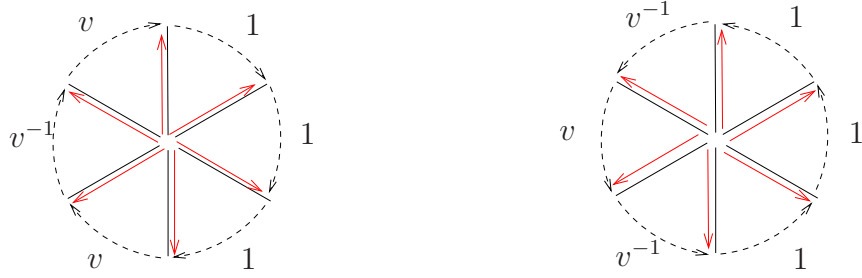


FIGURE 11

The product of weights assigned to the irreducible rotation between (p_k, j_k) and (p_l, j_l) following the clockwise (resp. anti-clockwise) order if $j_k > j_l$ (resp. if $j_l > j_k$) and setting $w(k, k) = 1$ is called the *weight between (p_k, j_k) and (p_l, j_l)* , denoted by $w(k, l)$.

Let $D^S = D_{P,J}^S$ be a disc with n boundary points, one distinguished as first (in green in the following figures), colored according to S , J and P . Let F_J be a flow on $W_{\lambda,Y}$ with boundary state J . The flow F_J determines a flow in \overline{W} , which we denote again by F_J . We represent this flow as a collection of arcs in D^S . Closed (resp. open) flow lines are drawn as floating oriented loops (resp. arcs) in D^S . An open flow line $f_{k,l}$ becomes an arc $d_{k,l}$ in D^S joining the boundary points k and l . We color these arcs according to the following rules. An arc $d_{k,l}$ in D^S is

- red, if $w(k, l) = v$;
- green, if $w(k, l) = v^{-1}$;
- black, otherwise.

Clockwise (resp. anti-clockwise) loops in are colored in red (resp. green). The collection of colored oriented arcs and loops associated to F_J obtained in this way will be called an *admissible configuration of D^S with state J* .

For an admissible configuration ℓ of D^S with state J we denote the number of

- red (resp. green) curves in ℓ by $R_J(\ell)$ (resp. $G_J(\ell)$);
- boundary vertices of D^S with state 0 and in position SE, S or SW by E_J ;
- boundary vertices of D^S either with state 1 and in position SE (resp. S^r), or with state -1 and in position SW (resp. S^l), by U_J .

Here, S^r (resp. S^l) indicates the boundary edges of D^S in position S and to the right (resp. left) of the first boundary point.

With these notions Theorem 3.7 can be restated as follows.

Theorem 4.2. *The invariant $[W_{\lambda,Y}] \in \text{Inv}(V^S)$ determined by the non-elliptic web $W_{\lambda,Y}$ expands as*

$$[W_{\lambda,Y}] = e_{J(W_{\lambda,Y})}^S + \sum_{J < J(W_{\lambda,Y})} v^{2U_J - E_J} \left(\sum_{\ell} v^{R_J(\ell) - G_J(\ell)} \right) e_J^S$$

where ℓ runs over all admissible configurations D_J^S with state $J < J(W_{\lambda,Y})$ in the lexicographic order.

Proof. We need to show that

$$v^{2U_J - E_J} \left(\sum_{\ell} v^{R_J(\ell) - G_J(\ell)} \right) = c(S, J(W_{\lambda,Y}), J)$$

when ℓ runs over all admissible configurations with boundary state $J < J(W_{\lambda,Y})$ and where $c(S, J(W_{\lambda,Y}), J)$ is the coefficient described in Theorem 3.7. For this, let F_{J_0} be a flow on $W_{\lambda,Y}$ with boundary state J_0 . This flow determines an admissible configuration ℓ_0 in D^S with boundary state J_0 . To determine the overall weight of F_{J_0} in $W_{\lambda,Y}$ we first notice that this weight agrees with the overall weight of F_{J_0} restricted to \overline{W} everywhere except (possibly) at both the U shaped boundary edges and the boundary edges with state 0. From the local weight chart of Figure 10 we deduce that for every flow with boundary state J_0 this difference is given by $v^{2U_{J_0} - E_{J_0}}$.

Next, the overall weight of an open flow line $f_{k,l}$ of F_{J_0} in \overline{W} is given by $w(k,l)$. In fact, by definition, the overall weight of $f_{k,l}$ is the product of local weights assigned to each trivalent vertex of $f_{k,l}$. Each such weight can equivalently be encoded in the angle of the corresponding irreducible rotation. Moreover, the endpoints k and l of \overline{W} with position and state (p_k, j_k) resp. (p_l, j_l) are linked through $f_{k,l}$ by a composition of distinct and successive irreducible rotations, after our assumption on $W_{\lambda,Y}$. This shows, that the weight of all open flow lines in F_{J_0} is given by the product of the weight of the corresponding arcs. In addition, closed flow lines always have weight v if oriented clockwise, resp. v^{-1} if oriented anti-clockwise. The weight of the various open and closed arcs can then be encoded in their color and the claim follows. \square

Orienting the flow lines on the left of Figure 12 gives rise to various flows on the non-elliptic web $W_{\lambda,Y}$. Its corresponding admissible configurations can be deduced from the figure on the right hand side.

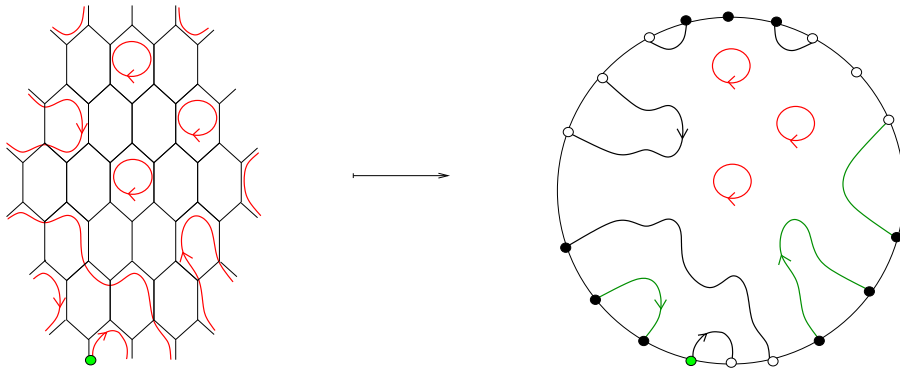


FIGURE 12

Moreover, both orientations of the open unoriented lines lead to the same coloring, hence have the same weight. Thus, Figure 12 leads to 16 different flows with overall weight 1 and different boundary state. It can be checked that none of the boundary states occurring matches the dominant lattice path $J(W_{\lambda,Y})$ of $W_{\lambda,Y}$. It follows that none of these flows is the leading flow of $W_{\lambda,Y}$.

5. LUSZTIG'S DUAL CANONICAL BASIS FOR $\text{Inv}(V)$

In this section we briefly recall the notion of Lusztig's dual canonical basis in $\text{Inv}(V)$. More details can be found in [33, 19, 20].

As before, for any signature $S = (s_1, s_2, \dots, s_n)$ and any state string $J = (j_1, j_2, \dots, j_n)$ let

$$e_J^S = e_{j_1}^{s_1} \otimes e_{j_2}^{s_2} \otimes \dots \otimes e_{j_n}^{s_n}.$$

Following Lusztig, for any tensor product V^S , there is a v -antilinear endomorphism $\Phi : V^S \otimes V^{S'} \rightarrow V^S \otimes V^{S'}$ defined inductively by the rule

$$\Phi(e^S \otimes e^{S'}) = \overline{\Theta}(\Phi^S(e^S) \otimes \Phi^{S'}(e^{S'}))$$

where $e^S \in V^S$ and $e^{S'} \in V^{S'}$ and $\overline{\Theta} \in U_q(\mathfrak{sl}_3(\mathbb{C})) \hat{\otimes} U_q(\mathfrak{sl}_3(\mathbb{C}))$ is the bar-conjugate of the quasi- R -matrix defined in a completion of $U_q(\mathfrak{sl}_3(\mathbb{C})) \otimes U_q(\mathfrak{sl}_3(\mathbb{C}))$. The action of Φ on V^\bullet is defined by $\Phi(\sum c_i e_i^\bullet) = \sum \bar{c}_i e_i^\bullet$ for $i \in \{1, 0, -1\}$ and all $c_i \in \mathbb{C}(v)$. Here $\bar{\cdot} : \mathbb{C}(v) \rightarrow \mathbb{C}(v)$ is the $\mathbb{C}(v)$ -algebra involution such that $\overline{v^n} = v^{-n}$ for all n . Similarly one defines the action of Φ on V° . It can be shown that Φ is well defined on all tensor powers of V° and V^\bullet and that $\Phi^2 = 1$. With this notion one defines Lusztig's dual canonical basis as follows.

Theorem 5.1 (Lusztig). *For any signature S and state string J there is a unique element $\ell_J^S \in V^S$ which is invariant under Φ and whose expansion in the tensor product basis has the form*

$$\ell_J^S = e_J^S + \sum_{J' < J} c(s, J, J') e_J^S,$$

with $c(s, J, J') \in v^{-1}\mathbb{Z}[v^{-1}]$.

A boundary state J has the *negative exponent property* if the coefficient $c(s, J, J')$ in Theorem 5.1 has the property that $c(s, J, J') \in v^{-1}\mathbb{Z}[v^{-1}]$ for every non-leading state $J' < J$.

The set $\{\ell_J^S\}$ is a $\mathbb{C}(v)$ -basis of V^S called the *dual canonical basis*. The subset of $\{\ell_J^S\}$ indexed by dominant paths is a $\mathbb{C}(v)$ -basis of $\text{Inv}(V^S)$, see [25] and references therein. By abuse of terminology, we will say that ℓ_J^S is dual canonical if it belongs to the dual canonical basis of $\text{Inv}(V^S)$. We will also write $\ell(W)$ instead of $\ell_{J(W)}^S$, when W is non-elliptic and S is clear from the context.

Remark 5.2. *The basis $\{\ell_J^S\}$ is dual to the canonical basis of V^S with respect to a bilinear form, see [19, §1.6] for details.*

Theorem 5.3. [25, Prop.2] *Every invariant of $\text{Inv}(V^S)$ defined by a non-elliptic web is invariant under Φ .*

Despite the fact that many non-elliptic web invariants of $\text{Inv}(V)$ are dual canonical basis elements and that Kuperberg's web bases share various properties with Lusztig's dual canonical bases, for example both bases are preserved under cyclic permutation of tensor factors, see [25, 27], these two bases are generally different from each other.

Theorem 5.4. [25, Thm.4] *Every basis web in $\text{Inv}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$, where each V_i is either V° or V^\bullet , is dual canonical when $n \leq 12$, except for a single basis web in*

$$\text{Inv}((V^\circ \otimes V^\circ \otimes V^\bullet \otimes V^\bullet)^{\otimes 3})$$

and its counterparts given by cyclic permutation of tensor factors.

Khovanov and Kuperberg proved this result providing explicitly a counterexample. To explain this consider again the family of Chebyshev polynomials of the second kind, $U_k([W], [B(W)])$ in $R_{a,b}(V)$ determined by the band product $\text{band}_k(W)$ described in Section 2. For each $k \in \mathbb{Z}_{\geq 0}$ let $[\text{Band}_k(W)]$ be the invariant defined by expanding the tensor diagram $\text{band}_k(W)$ in Kuperberg's web basis and unclaspings all endpoints. Then the counterexample of Theorem 5.4, in our notation, is given by $[\text{Band}_2(W)]$, described by the tensor diagram in Figure 3.

Remark 5.5. *From work of [36] it follows that the change of basis matrix from Kuperberg's web basis to the dual canonical basis of $\text{Inv}(V)$ is unitriangular. More details about this can be found in [36, Rem. 5.33].*

We next introduce an operation which will be useful in the following.

Definition 5.6. *Let D_1 and D_2 be two disjoint planar tensor diagrams which share endpoints. Assume that D_1 and D_2 do not cross or touch each other except at endpoints. We denote by $D_1 \cup D_2$ the tensor diagram obtained by superimposing D_1 and D_2 and then unclaspings all endpoints.*

The operation $[D_1 \cup D_2]$ can be extended by linearity to any invariant $[D_1] = \sum_i c_i [D_{1i}]$ in $\text{Inv}(V^S)$. Moreover, $[D_1 \cup D_2] = [D_2 \cup D_1]$ since D_1 and D_2 don't cross except at endpoints.

Remark 5.7. *In $R_{a,b}(V)$ the operation of Definition 5.6 represents the commutative multiplication of the corresponding invariants, that is: $[D_1 \cup D_2] = [D_1][D_2]$.*

The operation of Definition 5.6 preserves dual canonical elements in the following sense.

Proposition 5.8. *Let $i \in \{1, 2\}$ and let $[L_i] \in \text{Inv}(V^S)$ be two invariants for which $[L_1 \cup L_2]$ is defined. If each $[L_i]$ is dual canonical, then so is $[L_1 \cup L_2] \in \text{Inv}(V^S)$.*

Proof. Since the non-elliptic webs defining $[L_1]$ and $[L_2]$ are disjoint and non-crossing the negative-exponent property is preserved under the operations of Definition 5.6. \square

6. RED GRAPHS

Red graphs were introduced in [40]. They can be used to determine correction terms for a web invariant such that the result becomes dual canonical.

To implement this strategy, one associates to every non-elliptic web W of signature S a graded projective module $P_{J(W)}$ over the Khovanov-Kuperberg algebra K^S . Then one decompose $P_{J(W)}$ into a direct sum of graded projective indecomposable K^S -modules using red graphs, see [40, Thm. 3.10]. This direct sum of K^S -modules then decategorifies to a dual canonical element in $\text{Inv}(V^S)$ by [36, Thm. 5.31].

Let us begin this section by recalling definitions from [40, §.3]. In the following, let W be a web, W does not necessarily have to be non-elliptic.

Definition 6.1. *A red graph for W is a non-empty induced subgraph G of the dual graph of W , such that:*

- *the vertices of G correspond to some subsets of interior faces of W diffeomorphic to discs;*

- if f_1, f_2 and f_3 are adjacent faces of W sharing a vertex, then at least one face is not a vertex of G .

Let f be a vertex of a red graph G for W .

Definition 6.2. Half edges of W which are adjacent to f and which do not bound other faces of G are called gray half-edges of f in G .

Let the *external degree* of f , $\text{ed}(f)$ be the number of gray half-edges adjacent to f which do not bound f or another vertex of G .

Definition 6.3. Let o be an orientation of a red graph G of W . The level $i_o(f)$ of f of G is given by

$$i_o(f) := 2 - \frac{1}{2}\text{ed}(f) - |\{\text{edges of } G \text{ pointing to } f\}|.$$

The level of G is given by the sum of the levels of all vertices of G , or equivalently, by the formula:

$$I(G) = 2|V(G)| - |E(G)| - \frac{1}{2} \sum_{f \in V(G)} \text{ed}(f).$$

Definition 6.4. A red graph is *admissible* if one can choose an orientation o of G such that for every vertex f of G one has $i_o(f) \geq 0$. Such an orientation is called *fitting*. In addition, a red graph G for W is *exact* if $I(G) = 0$.

Lemma 6.5. Exact red graphs in non-elliptic webs have cycles.

Proof. If an red graph G is exact, $2|V(G)| - |E(G)| = \frac{1}{2} \sum_{f \in V(G)} \text{ed}(f)$. If G has no cycles, then G is a planar tree and $|E(G)| = |V(G)| - 1$. In particular, we then have

$$|V(G)| + 1 = \frac{1}{2} \sum_{f \in V(G)} \text{ed}(f).$$

On the other side, we know that the external degree of G is always an even number, see Remark after Def. 3.1 in [40]. In addition, since G has at least 3 vertices, see [40, Cor. 3.7], G has at least two terminal vertices with external degree greater then or equal to 4. It follows that

$$\frac{1}{2} \sum_{f \in V(G)} \text{ed}(f) \geq \frac{1}{2}(2(|V(G)| - 2) + 8) = |V(G)| + 2.$$

□

Definition 6.6. Let W be a web and let G be a red graph for W . A *pairing* of G is a partition of the gray half-edges of G into subsets of 2 gray half-edges adjacent to the same face of W , one pointing towards it, and the other pointing away from it. A red graph together with a pairing is called a *paired red graph*.

Definition 6.7. [40, Def. 3.16] Let W be a web and let G be a paired red graph for W . The G -reduction of W is the web W_G constructed as follows: To every face of W corresponding to a vertex of G

- remove all edges adjacent to this face;
- connect the gray half-edges of G according to the pairing.

Notice that the web W_G obtained as the G -reduction of W always has the same signature as W . Moreover, W_G is not necessarily non-elliptic, even when W is non-elliptic.

Important results linking the combinatorics of red graphs with the study of linear bases of $\text{Inv}(V)$ are [40, Thm. 3.10] together with [36, Thm. 5.31]. We summarize these results as follows:

Theorem 6.8. *Let $[W] \in \text{Inv}(V^S)$ be a web invariant of signature S and dominant lattice path $J(W)$. Let G be an exact paired red graph of W , then*

$$[W] = \ell(W) + \sum_{J(R) \neq J(W_{G_i})} k_{J(R)} \ell_{J(R)}^S + \sum_{J(W_{G_i}) \neq J(R)} k_{J(W_{G_i})} [W_{G_i}]$$

where W_{G_i} are non-elliptic webs arising from the G -reduction of W , $\ell(W) = \ell_{J(W)}^S$ and $\ell_{J(R)}^S$ are dual canonical, and all $k_{J(-)} \in \mathbb{C}(v)$.

Proof. We only need to show that the invariants $\sum_i k_i [W_{G_i}]$ belong to the above decomposition.

For this recall that to every non-elliptic web W a graded projective module $P_{J(W)}$ over the Khovanov-Kuperberg algebra K^S , up to isomorphism, can be associated, see [40, Def. 2.15].

If W has an exact paired red graph G , then $P_{J(W)}$ is decomposable and $P_{J(W)}$ has a direct factor isomorphic to a graded projective K^S -module associated to the web W_G obtained as a G -reduction of W , see Theorem 3.10 in [40]. We denote this direct factor of $P_{J(W)}$ by $P_{J(W_G)}$.

If W_G is elliptic and decomposes in Kuperberg's web basis as

$$W_G = n_1 W_{G_1} + \cdots + n_n W_{G_n},$$

for some $n_i \in \mathbb{C}(v)$ arising from skein relations, then $P_{J(W_G)}$ decomposes accordingly and we have

$$P_{J(W_G)} \cong P_{J(W_{G_1})} \oplus \cdots \oplus P_{J(W_{G_n})},$$

where P_- are projective graded K^S -modules (not necessarily indecomposable).

We then deduce:

$$\begin{aligned} P_{J(W)} &= Q_{J(W)} \oplus \bigoplus_{J(R) \neq J(W_{G_i})} Q_{J(R)}^{\oplus d(S, J(W), J(R))} \oplus \bigoplus_{J(W_{G_i}) \neq J(R)} P_{J(W_{G_i})}^{\oplus d(S, J(W), J(W_{G_i}))} \\ &= \ell(W) + \sum_{J(R) \neq J(W_{G_i})} k_{J(R)} \ell_{J(R)}^S + \sum_{J(W_{G_i}) \neq J(R)} k_{J(W_{G_i})} [W_{G_i}] \end{aligned}$$

where P_- are as before, Q_- are indecomposable graded projective K^S -modules, and $k_{J(-)} \in \mathbb{C}(v)$ are coefficients arising from the degree shifts. More precisely, for the first equality we decompose $P_{J(W)}$ into projective indecomposable graded K^S -modules together with the summands arising from G . The second equality follows from Theorem 5.31 in [36], since projective indecomposable graded K^S -modules are dual canonical up to multiplication by a constant. Moreover, by construction $P_{J(W_{G_i})}$ corresponds to $[W_{G_i}]$, up to multiplication with a constant. \square

Remark 6.9. *The rest of this work does not rely on the coefficients appearing in the previous result. For completeness, notice that these coefficients may be specifically determined with work of [36].*

Theorem 6.10. *Let $[W] \in \text{Inv}(V^S)$ be a web invariant. If W has no exact red graph, then $[W]$ is dual canonical.*

Proof. First notice that if G is an exact red graph for W then, by definition, the level of G is zero. With Theorem 3.11 in [40] (see also discussion after the statement) one deduces that the projective graded module corresponding to W is indecomposable. One then concludes with Theorem 5.31 in [36]. \square

7. CHEBYSHEV POLYNOMIALS AND LUSZTIG'S DUAL CANONICAL BASIS

Throughout the rest of the paper we fix the following set up. Let W be the simplest non-elliptic web with an internal face bounded by six edges. Let $[W \cup B \cup B] \in \text{Inv}(V)$ described by the non-elliptic web on the right of Figure 13. Let $[\text{Band}_5(W)] \in \text{Inv}(V)$, resp. $[\text{Thick}_5(W)] \in \text{Inv}(V)$, be described by unclasping all endpoints of the non-elliptic webs defining $[\text{band}_5(W)]$, resp. $[\text{thick}_5(W)]$ in $R_{a,b}(V)$.

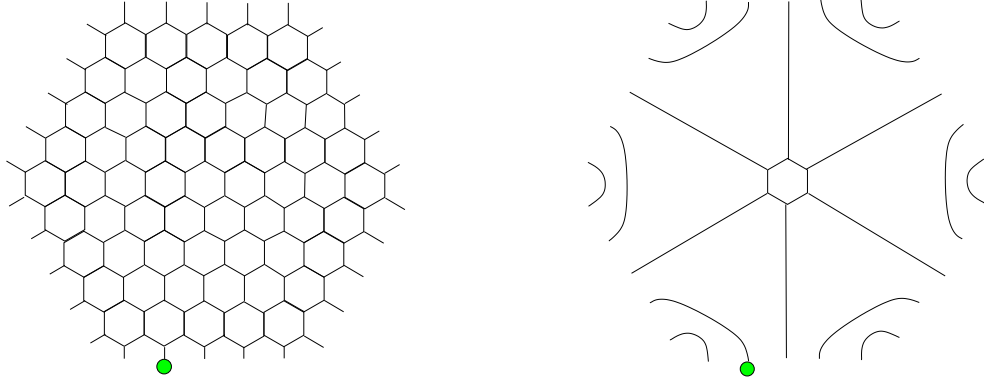


FIGURE 13. On the right an example of a web $D_1 \cup D_2 \cup D_3$ obtained from $D_1 = W$ and $D_2, D_3 = B$ is illustrated. On the left we find the non-elliptic web $\text{Thick}_5(W)$.

In Section 3 we saw that a unique state string, called the dominant lattice path or leading state, can be associated to $\text{Thick}_5(W)$ and $W \cup B \cup B$, after choosing a first boundary vertex. Here the first vertex is highlighted with a green dot in Figure 13. Notice that a different choice of initial point may give a different dominant lattice path. But by since dual canonical bases (as well as web bases) are preserved under the natural cyclic permutation operator, for the following, it is enough to determine one dominant lattice path.

Lemma 7.1.

- The dominant lattice path of $\text{Thick}_5(W)$, resp. of $W \cup B \cup B$, are:

$$\begin{aligned} J(\text{Thick}_5(W)) &= (11 \ 11111 \ 00111 \ 00000 \ -1-1000 \ -1-1-1-1-1 \ -1-1-1) \\ J(W \cup B \cup B) &= (11 \ -1-1111 \ -1-1111 \ -1-1011 \ -1-1011 \ -1-1-111 \ -1-1-1). \end{aligned}$$

- The signatures of $\text{Thick}_5(W)$, resp. $W \cup B \cup B$, are both equal to:

$$S^5 = (\bullet \bullet \circ \circ \circ \circ \circ \bullet \bullet \bullet \bullet \bullet \circ \circ \circ \circ \circ \bullet \bullet \bullet \bullet \bullet \circ \circ \circ \circ \circ \bullet \bullet \bullet).$$

- The signature of $\text{Thick}_3(W)$ is:

$$S^3 = (\bullet \circ \circ \circ \bullet \bullet \bullet \circ \circ \circ \bullet \bullet \bullet \circ \circ \circ \bullet \bullet).$$

Proof. The first part of the claim can be computed with the minimal cut algorithm, described in Section 3.1, on the non-elliptic webs $\text{Thick}_5(W)$ and $W \cup B \cup B$ specifying the first boundary vertex as in Figure 13. The signatures are cyclic words that can be read off directly from the boundary points of the corresponding webs in the clockwise order. \square

Lemma 7.2. *The invariant $[\text{Thick}_5(W)] \in \text{Inv}(V^{S^5})$ decomposes in the tensor product basis as*

$$[\text{Thick}_5(W)] = e_{J(\text{Thick}_5(W))}^{S^5} + 6e_{J(W \cup B \cup B)}^{S^5} + \sum_{J' < J(\text{Thick}_5(W))} c(S^5, J(\text{Thick}_5(W)), J') e_{J'}^{S^5}$$

for $c(S^5, J(\text{Thick}_5(W)), J') \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

Proof. From Theorem 3.7 we know that $c(S^5, J(\text{Thick}_5(W)), J') \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$. Hence we only need to show that

$$c(S^5, J(\text{Thick}_5(W)), J(W \cup B \cup B)) \geq 6.$$

For this it is enough to exhibit 6 different flows on $\text{Thick}_5(W)$ with boundary state $J(W \cup B \cup B)$ and of weight 1. These flows are illustrated in Figure 14.

To check that these flows have weight 1 we observe that $\text{Thick}_5(W)$ satisfies the assumptions of Section 4, since all interior faces of $\text{Thick}_5(W)$ are hexagons. Then we know from Theorem 4.2 that the overall weight of each flow is given by

$$v^{2U-E} v^{R-G}$$

for R (resp. G) the number of red (resp. green) flow lines in each given flow. Moreover, we observe that here

$$2U - E = 2U = 8,$$

as there are 2 half-edges in position SE and with state 1 together with 2 half-edges in position SW and state -1.

The color of each flow line is determined by the position and state of the boundary edges at the start and end of the flow line, see Theorem 4.2.

The inequality then follows observing that weights in state sums never cancel. \square

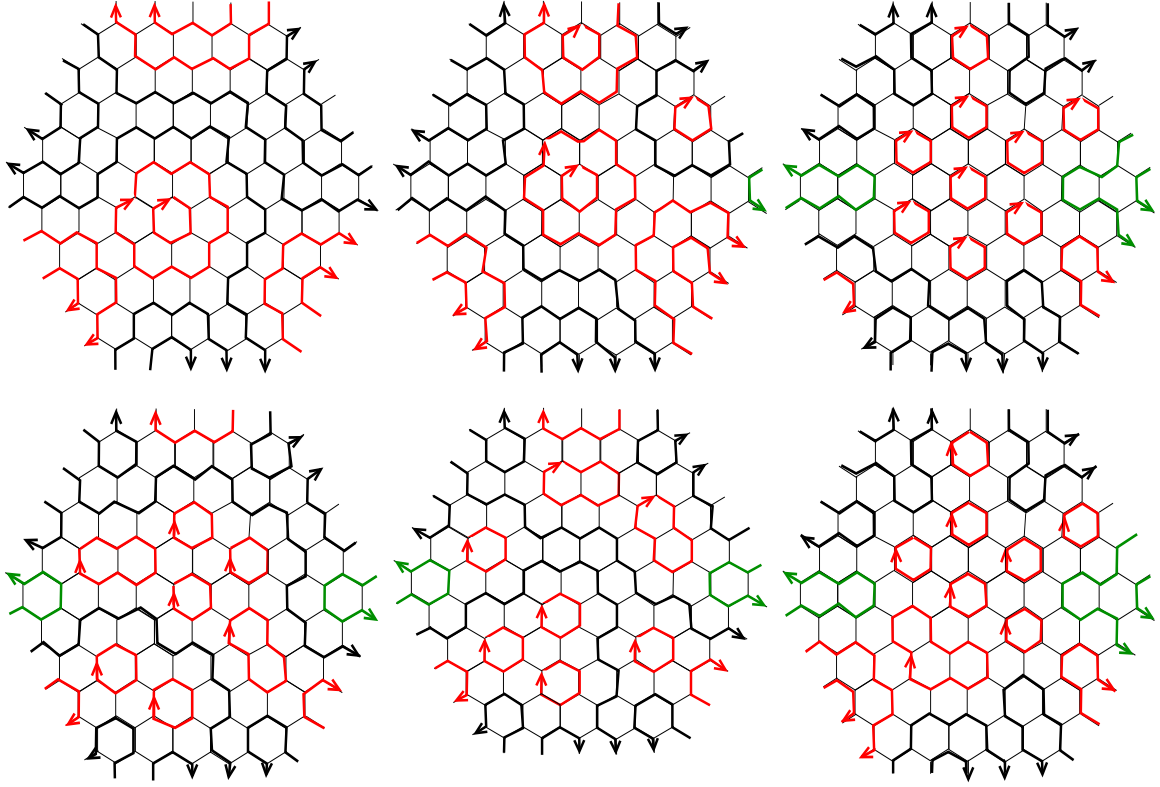


FIGURE 14. Different flows with boundary state $J(W \cup B \cup B)$ and positive weight on $\text{Thick}_5(W)$.

With Lemma 7.2 one also deduces that Kuperberg's basis element $[\text{Thick}_5(W)] \in \text{Inv}(V^{S^5})$ does not satisfy the negative exponent property. Hence $[\text{Thick}_5(W)]$ is not dual canonical, by Theorem 5.1. Similarly, one can show that $[\text{Thick}_3(W)] \in \text{Inv}(V^{S^3})$ is not dual canonical.

Lemma 7.3. *For $k \in \{3, 5\}$, there are dual canonical basis elements $\ell(\text{Thick}_k(W)) \in \text{Inv}(V^{S^k})$, which decompose in Kuperberg's web basis as:*

$$\ell(\text{Thick}_3(W)) = [\text{Thick}_3(W)] - a_1[W \cup B] - \sum_j b_j[L_j]$$

$$\ell(\text{Thick}_5(W)) = [\text{Thick}_5(W)] - c_1[\text{Thick}_3(W) \cup B] - c_2[W \cup B \cup B] - \sum_k d_k[L_k]$$

where all $[L_-]$ are dual canonical web invariants determined by non-elliptic webs with a Y at the boundary and where $a_1, b_j, c_1, c_2, d_k \in \mathbb{C}(v)$.

Proof. First we identify all possible exact paired red graphs in $\text{Thick}_k(W)$, $k \in \{3, 5\}$. For this we notice that to be exact such a graph must have at least one cycle and each such cycle has at least 6 vertices by [40, Prop. 4. 15.]. Without loss of generalities, here it is enough to consider exact red graphs consisting only of one cycle and with no additional tree attached to the cycle. For these exact red graphs there is only one possible paring, see Remark after [40, Cor. 3.7].

Second, for all such exact paired red graphs G , we now analyze all possible G -reductions of $\text{Thick}_k(W)$, $k \in \{3, 5\}$, occurring. To do this we check cases, that is we vary both the

position of the cycle as well as we augment the number of vertices of the cycle of G . In this way, we notice that all G -reductions of $\text{Thick}_k(W)$, $k \in \{3, 5\}$, split into two classes: those with a Y piece at the boundary and those without. It can be checked that the latter are precisely $\text{Thick}_3(W) \cup B$ and $W \cup B \cup B$, obtained as G -reductions of $\text{Thick}_5(W)$, resp. $W \cup B$, obtained as G -reduction of $\text{Thick}_3(W)$. All other webs arising have a Y piece at the boundary, and we denote them by L_- . Notice that these Y pieces at the boundary of L_- remain after solving the web using skein relations.

Observe that the non-elliptic webs $[L_-]$ with a Y piece at the boundary may be decomposed further iterating the above procedure. As the Y at the boundary remains unchanged, this procedure never yields $W \cup B$, resp. $\text{Thick}_3(W) \cup B$ or $W \cup B \cup B$.

Finally, the terms $[L_-]$ are dual canonical because the non-elliptic webs defining these invariants have been decomposed until they don't contain any exact red graph and are dual canonical by Theorem 6.10. \square

Corollary 7.4. *For $k \in \{3, 5\}$ the dual canonical basis element $\ell(\text{Thick}_k(W))$ expands in Kuperberg's web basis as $[\text{Band}_k(W)]$, modulo invariants defined by tensor diagrams with a Y at the boundary.* \square

Proof of Proposition 1.5. We assume that $[\text{band}_3(W)]$, resp. $[\text{Band}_3(W)]$, belong to the specialization of Lusztig's dual canonical basis at $q = 1$. We show that in this case there is no dual canonical basis element in $\text{Inv}(V^{S^5})$ with integer coefficients that specializes to $[\text{Band}_5(W)]$, when $q \rightarrow 1$.

We proceed by contradiction, assuming that $\ell(\text{Thick}_5(W)) \in \text{Inv}(V^{S^5})$, in the expansion given in Lemma 7.3, has integer coefficients and is such that both $c_1 = 4$ and $c_2 = -3$.

Consider the two invariants $[B]$, defined by the non-elliptic web consisting of a sextuple of U shaped tensor diagrams, and $\ell(\text{Thick}_3(W))$. Since B is homotopic to the boundary of the disc containing the summands of $\ell(\text{Thick}_3(W))$ Definition 5.6 is satisfied and we can consider $[\ell(\text{Thick}_3(W)) \cup B] \in \text{Inv}(V^{S^5})$.

Next, rewrite $\ell(\text{Thick}_5(W))$, using Lemma 7.3, as follows:

$$\begin{aligned} \ell(\text{Thick}_5(W)) &= [\text{Thick}_5(W)] - c_2[W \cup B \cup B] - \sum_k d_k[L_k] - c_1[\text{Thick}_3(W) \cup B] \\ &= [\text{Thick}_5(W)] - c_2[W \cup B \cup B] - \sum_k d_k[L_k] \\ &\quad - c_1 \left([\ell(\text{Thick}_3(W)) \cup B] + a_1[W \cup B \cup B] + \sum_j b_j[L_j \cup B] \right) \\ &= [\text{Thick}_5(W)] - (c_1 a_1 + c_2)[W \cup B \cup B] - \sum_k d_k[L_k] \\ &\quad - c_1[\ell(\text{Thick}_3(W)) \cup B] - \sum_j c_1 b_j[L_j \cup B]. \end{aligned}$$

Using Lemma 5.8, observe that since $[W]$, $[B]$ and $[L_j]$, resp. $[L_k]$, (by Lemma 7.3) are dual canonical, also $[L_j \cup B]$ and $[W \cup B \cup B]$ are dual canonical. Similarly, since $\ell(\text{Thick}_3(W))$ is dual canonical by assumption the same is true for $[\ell(\text{Thick}_3(W)) \cup B]$.

Moreover, the leading states of $[L_j \cup B]$, $[L_k]$ and $[\ell(\text{Thick}_3(W)) \cup B]$ are $J(L_j \cup B)$, $J(L_k)$ resp. $J(\text{Thick}_3(W) \cup B)$ and they are all different then the leading state of $W \cup B \cup B$, which was denoted by $J(W \cup B \cup B)$.

To conclude observe that $\ell(\text{Thick}_5(W))$ has the non-negative exponent property, implies that

$$c(S^5, J(\text{Thick}_5(W)), J(W \cup B \cup B)) = (c_1 a_1 + c_2).$$

But this is impossible, since $(c_1 a_1 + c_2) = 5$ and we proved in Lemma 7.2 that

$$c(S^5, J(\text{Thick}_5(W)), J(W \cup B \cup B)) \geq 6.$$

□

A few remarks are in order.

Remark 7.5. *In the statement of Proposition 1.5, if one omits the assumption on integer coefficients, one allows in the proof above that $c(S^5, J(\text{Thick}_5(W)), J(W \cup B \cup B)) \in \mathbb{Z}[v, v^{-1}]$. In this case, one would have to consider all flows with boundary state $J(W \cup B \cup B)$ smaller than $J(L_k)$ and $J(L_j \cup B)$, for the lexicographic order, to prove the corresponding statement.*

Remark 7.6. *It is possible to extend the band operation of $[W]$ in $R_{a,b}(V)$ to the non-commutative setting of $\text{Inv}(V)$ by vertically superimposing copies of W , bundling together endpoints, and solving the crossings using the quantum skein relations of [27]. It is not difficult to see that, by doing so and expressing the corresponding invariant in Kuperberg's basis, one obtains again the recursions of Theorem 1.2, up to an overall scalar given by a power of the indeterminate v .*

Remark 7.7. *There are values of a, b such that the cluster algebra structure on $R_{a,b}(V)$ contains a sub-cluster algebra of affine type $A_1^{(1)}$, see [14]. It can then be shown, that the invariant $[W]$, defined by the minimal non-elliptic single-cycle web W , can be expressed as $[W] = z_0 z_3 - z_1 z_2 \in R_{a,b}(V)$ where all z_i are cluster variables belonging to the two clusters $\{z_0, z_1\}$ and $\{z_2, z_3\}$ of $R_{a,b}(V)$, up to coefficient variables. This description of $[W]$ matches the topological description of the variable z parametrized by the closed loop in the annulus with one marked point on each boundary component.*

Since $R_{a,b}(V)$ often carries simultaneously several cluster algebra structures, it is not clear how the decomposition of $[W]$ in the last remark generalizes to $\text{Inv}(V)$. Moreover, it is not clear which diagrammatic representation of the product rule should be considered and how to match the rule of the coefficient variables for the given cluster algebra structure.

Despite this difficulties, and in light of the results of [3, 28, 9], a possible conjecture to formulate appears to be the following:

Conjecture 7.8. *There is an invariant $[L] \in R_{a,b}(V)$ with the following properties:*

- *$[L]$ is defined by a single tensor diagram L (not necessarily planar) with a single internal cycle of lengths 6 or greater.*
- *There is a quantum cluster subalgebra of Kronecker type in $\text{Inv}(V)$ such that $[L]$ decomposes as $v z_0 z_3 - v^3 z_1 z_2$ for z_0, z_1, z_2, z_3 quantum cluster variables in the two clusters $\{z_0, z_1\}$, and $\{z_2, z_3\}$.*
- *The band operation of $[L]$ in $R_{a,b}(V)$ is defined.*

Then, for all $k \geq 1$ there are dual canonical basis elements of $\text{Inv}(V)$ which specialize to $[\text{Band}_k(L)]$, after $q \rightarrow 1$ and restitution.

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